MATHEMATICAL

AND

PHYSICAL PAPERS

BY THE LATE

SIR GEORGE GABRIEL STOKES, BART.,
Sc.D., LL.D., D.C.L., PAST PRES. R.S.,
KT PRUSSIAN ORDER *POUR LE MÉRITE*, FOR. ASSOC. INSTITUTE OF FRANCE, *ETC.*
MASTER OF PEMBROKE COLLEGE AND LUCASIAN PROFESSOR OF MATHEMATICS
IN THE UNIVERSITY OF CAMBRIDGE.

Reprinted from the Original Journals and Transactions,
with brief Historical Notes and References.

VOL. IV.

CAMBRIDGE:

Cambridge:
PRINTED BY J. AND C. F. CLAY,
AT THE UNIVERSITY PRESS.

PREFACE.

IN the preface to the third volume Sir George Stokes expressed his intention, should life and health last, to complete the republication of his Scientific Papers without delay. These conditions were not destined to be fulfilled; and the task of completing this memorial of his genius has fallen to other hands.

This fourth volume includes the Papers that were published between 1853 and 1876. In accordance with the plan followed in the previous volumes, only memoirs and notes involving distinct additions to scientific knowledge have been included; this restriction has however had little effect except in the omission of some addresses, extracts from which will be collected together in another place. The texts of the papers have been reproduced as they originally appeared, with the exception that obvious misprints have been corrected. A few historical and elucidatory footnotes, those within square brackets, have been added by the Editor.

Advice has been received, in the selection and treatment of the material, from Lord Kelvin and Lord Rayleigh, who have seen most of the proof sheets. Acknowledgment is also due to Prof. Liveing who has very kindly examined the papers dealing with

The remaining papers, together with a biographical notice which is being prepared for the Royal Society by Lord Rayleigh, will appear in a fifth volume. It is hoped that it will be possible to arrange a selection from Sir George Stokes' scientific correspondence, for publication either there or in a separate form: some extracts from it, relating to the contents of the present volume, are here printed in an Appendix.

The portrait prefixed to this volume is taken from an oil painting made in 1874 by Mr Lowes Dickinson, which belongs to Pembroke College.

J. LARMOR.

St John's College, Cambridge,
January, 1904.

CONTENTS.

The remaining papers, together with a biographical notice which is being prepared for the Royal Society by Lord Rayleigh, will appear in a fifth volume. It is hoped that it will be possible to arrange a selection from Sir George Stokes' scientific correspondence, for publication either there or in a separate form: some extracts from it, relating to the contents of the present volume, are here printed in an Appendix.

The portrait prefixed to this volume is taken from an oil painting made in 1874 by Mr Lowes Dickinson, which belongs to Pembroke College.

J. LARMOR.

St John's College, Cambridge,
January, 1904.

CONTENTS.

MATHEMATICAL AND PHYSICAL PAPERS.

On the Change of Refrangibility of Light.—No. II.

[From the *Philosophical Transactions* for 1853, pp. 385—396.
Received *June* 16, read *June* 26, 1853.]

The chief object of the present communication is to describe a mode of observation, which occurred to me after the publication of my former paper, which is so convenient, and at the same time so delicate, as to supersede for many purposes methods requiring the use of sun-light. On account of the easiness of the new method, the cheapness of the small quantity of apparatus required, and above all, on account of its rendering the observer independent of the state of the weather, it might be immediately employed by chemists in discriminating between different substances.

I have taken the present opportunity of mentioning some other matters connected with the subject of these researches. The articles are numbered in continuation of those of the former paper *.

Method of observing by the use of Absorbing Media.

241. Conceive that we had the power of producing at will media which should be perfectly opaque with regard to rays belonging to any desired regions of the spectrum, from the extreme red to the most refrangible invisible rays, and perfectly transparent with regard to the remainder. Imagine two such media prepared, of which the second was opaque with regard to those rays of the *visible* spectrum with regard to which the first was transparent, and *vice versâ*. It is clear that if both

from the eye, and placed so as to intercept all the rays which fell on certain objects, which were then viewed through the second, provided the objects did nothing more than reflect, refract, scatter, or absorb the incident rays. But if any of the objects had the property of emitting rays of one refrangibility under the influence of rays of another, it might happen that some of the rays so emitted were capable of passing through the second medium, in which case the object would appear luminous in a dark field.

242. Let us consider now how the media must be arranged so as to bring out to the utmost the sensibility of a given substance. To take a particular instance, suppose the substance to be glass coloured by uranium. In this case the sensibility of the medium begins, with almost absolute abruptness, near the fixed line *b* of Fraunhofer, and continues from thence onwards. The dispersed light has the same, or at least almost rigorously the same, composition throughout, and consists exclusively of rays less refrangible than *b*. Consequently, we should have to prepare a first medium which was opaque with regard to the visible rays less refrangible than *b*, and transparent with respect to the rays, whether visible or invisible, more refrangible, and a second medium complementary to the former in the manner described in the preceding article. If the pair of media were still strictly complementary in this manner, but the point of the spectrum at which the transparency of the first medium began and that of the second ended were situated at some distance from *b*, the sensibility of the glass would be exhibited as before, only the maximum effect would not be produced, on account of the absorption of a portion either of the active or of the dispersed rays, according as the point in question was situated above or below *b*.

Now, although the commencement of the sensibility of canary glass is unusually abrupt, it generally happens that the sensibility of a medium, or at least the main part of it, comes on with great rapidity, and lasts throughout the rest of the spectrum, though frequently it is most considerable in a region extending not very greatly beyond the point where it commenced. In those cases in

Hence, if we could prepare absorbing media at pleasure, we should get ready for general use in these observations a few pairs of media complementary in the particular manner already described, but having the points of the spectrum at which the transparency of the first medium commenced and that of the second ended different in different pairs, situated say in the yellow for one pair, in the blue for another, in the extreme violet for a third.

243. It is not of course possible to prepare media in this manner at pleasure, and all we can do is to select from among those which occur in nature. Nevertheless it is useful, as a guide in the selection, to consider what constitutes the ideal perfection of absorbing media for this particular purpose. But before proceeding to mention the media which I have found convenient, I will describe the arrangement which I have adopted for admitting the light.

A hole was cut in the window-shutter of a darkened room, and through this the light of the clouds and external objects entered in all directions. The diameter of the hole was four inches, and it might perhaps have been still larger with advantage. A small shelf, blackened on the top, which could be screwed on to the shutter immediately underneath the hole, served to support the objects to be examined, as well as the first absorbing medium. This, with a few coloured glasses, forms all the apparatus which it is absolutely necessary to employ, though for the sake of some experiments it is well to be provided also with a small tablet of white porcelain, and an ordinary prism, and likewise with one or two vessels for holding fluids.

244. In the observation, the first medium is placed resting on the shelf so as to cover the hole; the object is placed on the shelf immediately in front of the hole; the second medium is held anywhere between the eyes and the object. As it is not possible to obtain media which are strictly complementary, it will happen that a certain quantity of light is capable of passing through both media. This might no doubt be greatly reduced by increasing the

Accordingly, it might sometimes be doubtful whether the illumination perceived on the object were due merely to scattered light which had passed through both media, or to really "degraded" light *. To remove all doubt, it is generally sufficient to transfer the second medium from before the eyes to the front of the hole. The light merely scattered by the object will necessarily be the same as before, if the room be free from stray light; and even if there be a little stray light, the illumination, so far as it is concerned, will be increased instead of diminished; whereas if the illumination previously observed were due to fluorescence, and the media were properly chosen, the object which before was luminous will now be comparatively dark.

Sometimes, in the case of substances which have only a low degree of sensibility, it is better to leave the second medium in front of the eyes, and use a third medium, which is held alternately in the path of the rays incident on the object and between the

* This term, which was suggested to me by my friend Prof. Thomson, appears to me highly significant. The expression *degradation of light* might be substituted with advantage for *true internal dispersion* to designate the general phenomenon; but it is perhaps a little too wide in its signification, and might be taken to include phosphorescence (if indeed in this case the refrangibility be really always lowered), as well as the emission of non-luminous radiant heat by a body which had been exposed to the red rays of the spectrum. As to the term *internal dispersion*, though I employed it, following Sir David Brewster, I confess I never liked it. It seems especially awkward when applied to a washed paper or dyed cloth; it was adopted at a time when the phenomenon was confounded with opalescence; and, so far as it implies theoretical notions at all, it seems rather to point to a theory now no longer tenable: I allude to the theory of suspended particles. Indeed, this theory, as it seems to me, ceased to be tenable as soon as Sir John Herschel had discovered the peculiar analysis of light connected with epipolic dispersion, and Sir David Brewster had connected the phenomenon with internal dispersion, so far at least as the common appearance of a continuous and coloured dispersed beam formed a connexion. The expression *dependent emission* is awkward, but would be significant, because the light is emitted in the manner of self-luminous bodies, but only in dependence upon the active rays, and so long as the body is under their influence. In this respect the phenomenon differs notably from phosphorescence. It is quite conceivable that a continuous transition may hereafter be traced by experiment from the one phenomenon to the other, but no such transition has yet been traced, nor is it by any means certain that the phenomena are not radically distinct. On this account it would, I conceive, be highly objectionable to call true

object and the eyes. Such a medium, though not at all necessary, may be used also in the case of highly sensitive substances, for the sake of varying the experiment and rendering the result more striking.

As it will be convenient to have names for the media fulfilling these different offices, I will call the first medium, or that with which the whole is covered, the *principal absorbent*, the second medium the *complementary absorbent*, and the third medium, when such is employed, the *transfer medium*. For the transfer medium we may choose a medium of the same nature as the complementary absorbent, but paler. This is perhaps the best kind to employ in the methodical examination of various substances; but if the object of the observer be merely to illustrate the phenomena of the change of refrangibility of light, he may vary the experiments by using other media.

245. I have hitherto spoken only of the increase of illumination due to the sensibility of the substance under examination. But independently of illumination, the colour of the emitted light affords in most cases a ready means of detecting fluorescence. Thus, suppose the principal absorbent to transmit no visible rays but deep blue and violet, and the substance examined to appear, when viewed through the complementary absorbent, of a bright orange colour. Since no combination of the rays transmitted by the principal absorbent can make an orange, we may instantly conclude that the substance is sensitive. However, I do not consider it safe, at least for a beginner, to trust very much to *absolute* colour, for few who have not been used to optical experiments can be aware to what an extent the eye under certain circumstances is liable to be deceived. The *relative* colour of two objects seen at the same time under similar circumstances may usually be judged of safely enough; that is, of two such colours it may be possible to say with certainty that one inclines more to blue or to red than the other. Of course in many cases the change of colour is so great that there can be no mistake; still I think it a safe rule for a person employing these modes o

246. If it be desired to view the object isolated as much as possible, it may be placed directly on the shelf, or better still, on black velvet. But it is generally preferable to have for comparison a standard object which reflects freely the visible rays, of whatever kind, incident upon it, and does not possess any sensible degree of fluorescence. It is in this way that the white porcelain tablet is useful; and in observing, I generally place the tablet on the shelf and the object on it. A white plate would answer, but a tablet is better, on account both of its shape and of the comparative dullness of its surface. It is true that the tablet used exhibited a very sensible amount of fluorescence when examined in a linear spectrum formed by a quartz train; still the effect was so small, and so much of it was due to those highly refrangible rays which are stopped by glass, that for practical purposes the tablet may be regarded as insensible. However, an observer is not obliged either to assume that all tablets are alike, or to apply to the particular tablet which he proposes to use, methods of observation requiring the use of apparatus which he is not supposed to possess. The methods of observation described in the present paper are complete in themselves; the observer has it in his power to test for himself the tablet he proposes to employ; and he is bound to do so before taking it for a standard of comparison. It may easily be tested by means of a prism, as will be explained presently.

247. The following are the combinations of media which I have chiefly employed:—

First Combination.—In this combination the principal absorbent is a glass coloured deep violet by manganese with a little cobalt, or else a glass coloured deep purple by manganese alone, combined with a rather pale blue glass coloured by cobalt, or with a deeper blue glass in case the day be bright. It is very easy to tell by means of a prism, with candle-light, whether a purple glass contains any sensible quantity of cobalt, on account of the very peculiar mode of absorption which is characteristic of this metal. In the examination of substances by this combination no

SECOND COMBINATION.—In this case the principal absorbent is a solution of the ammoniaco-sulphate of copper, employed in such thickness as to give a deep blue. In my experiments the fluid was contained in a cell with parallel sides of glass, which was closed at the top for greater convenience; but a very broad flat bottle would answer as well, because in the case of the principal absorbent the regularity of refraction of rays across it is of no consequence. Such a bottle, however, would have to be ordered expressly. The complementary absorbent in this combination is a yellow glass coloured by silver, and slightly overburnt. These glasses, as commonly prepared, are opaque with regard to most of the violet, but become transparent again with regard to the invisible rays beyond; and, in the case of a pale glass, the commencing transparency in the extreme violet may even be perceived by means of light received directly into the eye. I have got a glass of a pretty deep orange-yellow colour, which is more transparent than common window-glass with regard to rays of such high refrangibility as to be situated near the end of the region of the solar spectrum which it requires a quartz train to show. But when too much heat is used in the preparation, the glass acquires, on the interior of the coloured face, a delicate blue appearance, having a good deal of the aspect of a solution of sulphate of quinine, though it has in reality nothing to do with fluorescence; and in this state the glass is nearly opaque with regard to the invisible rays of the solar spectrum beyond the violet, though it still transmits a few among those which are nearly the most refrangible. Of course, if the complementary absorbent were always left in its position between the eyes and the object, its transparency or opacity with regard to invisible rays would be a matter of indifference; but as it is desirable that its transference from that position to the front of the hole should produce as much difference as possible, it is important that it should be opaque, or nearly so, with regard to the ultra-violet rays transmitted by the principal absorbent. Hence one of these slightly over-burnt glasses should be selected for the present purpose, and such are very commonly met with.

FOURTH COMBINATION.—In this the principal absorbent is a solution of nitrate of copper, and the complementary absorbent a light red or deep orange glass.

248. In the first combination the darkness is tolerably complete without the use of any complementary absorbent, since no visible rays are transmitted except violet and some extreme red. The latter are no inconvenience, but rather help to set off the tint of the light due to fluorescence. This is, I think, the best combination to employ when the fluorescent light is blue, or at least deep blue; because in that case much of the light is lost by absorption in the yellow glass employed in the second combination. It has the advantage, too, of allowing the fluorescent light to enter the eye without being modified by absorption. Nevertheless no correct estimate can be formed of the absolute colour of the fluorescent light without making very great allowance for the effects of contrast, especially when the body, instead of being isolated as far as possible, is placed on the porcelain tablet.

The second combination is on the whole the most powerful. The media in this case make a very fair approach towards the ideal perfection explained in Art. 242. The darkness is so far complete, or else may easily be made so by increasing a little the strength of either absorbent, that if the tablet be written on with ink, and placed on the shelf between the media, the writing cannot be read. It forms a striking experiment, after having treated the tablet in this manner, to introduce between the media a piece of canary glass or a similar medium. The glass is not only luminous itself, but it emits so much light as to illuminate the whole tablet, so that the writing is instantly visible. In those cases in which the fluorescent light is yellow, orange, or red, it is shown a good deal more strongly by this combination than by the preceding.

The third combination is applicable to the same cases as the second. The blue glass answers extremely well, but is not quite so good as the blue fluid. The

person who does not happen to have a vessel of the proper shape for holding the fluid.

In the second and third combinations the point of the spectrum at which the transparency of the principal absorbent begins, and that of the complementary absorbent ends, or rather the point which most nearly possesses this character, is situated in the blue. Thinking that the fluorescence of those substances which emit light of low refrangibility might be better brought out if this point was situated lower down in the spectrum, I tried the fourth combination. In this case the media have very fairly the required complementary character; the darkness is pretty complete, and the fluorescence of scarlet cloth and similar substances is very well exhibited. However, the effect in these cases is shown so well by the second combination, that, except it be for the sake of varying the experiment, I do not think it worth while to employ the fourth combination, more especially as it has the disadvantage of leaving the observer in doubt whether the red or orange light perceived constitutes the whole of the fluorescent light, or only that part of it which alone has been able to get through the complementary absorbent.

249. The mode of observation may be altered in various ways which afford pleasing illustrations of the theory, though in the regular examination of a set of substances it is best to proceed in a more methodical manner. Thus, if nothing but a violet or blue glass or a blue fluid be used as a principal absorbent, and the substances under examination be highly sensitive, their appearance will be remarkably changed if the coloured medium be transferred from before the hole to before the eyes. Again, if the complementary absorbent be made to exchange places with the principal absorbent, the result will be similar, although the very same media are merely interposed in different parts of the compound path of the light from the clouds to the eye. If a transfer medium be employed, and it be, as has hitherto been supposed, of the same general nature as the complementary absorbent, it will not produce much effect when it is interposed between the object and the eyes, but when it is placed in the path of the rays incident on the

will be just the reverse. This is strikingly shown in the case of a substance, which, like scarlet cloth, emits a red fluorescent light, by taking for a transfer medium a solution of nitrate of copper, and in the case of turmeric paper or yellow uranite, by taking the same solution, or else a blue glass. In the case of the two substances last mentioned, if we take for a transfer medium a red solution of mineral chameleon, diluted so as to be merely pink, the intensity of the light emitted will, under certain conditions, be not much different in the two positions of the medium, because a portion of the active rays in one position and a portion of the degraded rays in the other will be absorbed; but the colour of the portion of the emitted light which reaches the eye will be altogether different in the two positions of the transfer medium.

Mode of observing by means of a Prism.

250. In this method no absorbing medium is required except the principal absorbent. The white tablet being laid on the shelf, a slit is first held in such a position as to be seen projected against the sky, and the light thus coming directly into the eye, after having passed through the principal absorbent, is analysed by a prism held in the other hand. The slit is now held so that the tablet is seen through it, and the light coming from the tablet is analysed. It will be found that the spectrum seen in the first instance is faithfully reproduced, being merely less luminous, as must necessarily happen. At least, this was the case in those tablets which I have examined; and in this way each observer ought to test for himself the tablet he proposes to employ. After having been thus tested, the tablet may be used as a standard of comparison.

Suppose now that it is wished to examine a slip of turmeric paper, or a riband, or other similar object. The object is laid on the tablet, and the slit held immediately in front of it, in such a manner that one part, suppose the central portion, of the slit is seen projected on the object, and the remainder on the tablet. The light coming through the slit is then analysed by the prism,

Occasionally in these observations a blue glass is preferable to a solution of the ammoniaco-sulphate of copper, because the extreme red and the greenish yellow bands transmitted by the glass, while too faint to interfere with the fluorescent light, are useful as points of reference.

251. The general appearance of the spectrum in this mode of observation may be gathered from the accompanying figures, of which the first represents turmeric paper seen under the blue glass, and the second represents a mass of crystals of nitrate of uranium seen under the copper solution. In fig. 1, *RR'*, *YY'*

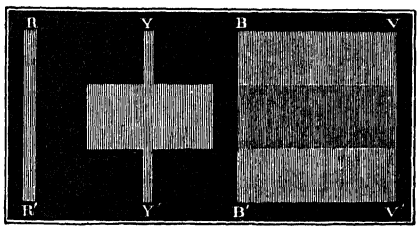
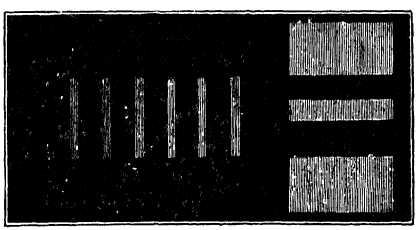


Fig. 1.

are the red and yellow bands transmitted by the glass, which are seen equally in the light scattered by the tablet and that scattered by the paper. *BVB'V'* is the blue and violet light transmitted by the glass. Of this a considerable portion, especially in the more refrangible part, is absorbed by the turmeric paper, which on the other hand emits a quantity of red, yellow, and green light, not found among the incident rays. Fig. 2 sufficiently explains itself. In this case the fluorescent



the mass of crystals, except where a crystalline face happens to be situated in such a position as to reflect the light of the sky into the eye, as represented in the figure. In the case of a substance so highly sensitive as nitrate of uranium, and which does not, like a slip of paper, lie flat on the tablet, the spectrum of the fluorescent light in reality extends, at least on the side next the window, though with less intensity, to some distance beyond the part of the slit which corresponds to the object, because the tablet is lighted up by the rays emitted by the object; but this is not represented in the figure.

252. The mode of using the prism just explained is that by which the phenomenon of the change of refrangibility is most strikingly illustrated; but in the actual examination of substances the chief use of the prism is to determine, in the case of substances which are sufficiently sensitive to admit with advantage of such a mode of observation, the composition of the fluorescent light. For this purpose it is often better to isolate the object by placing it on black velvet. This is especially the case with very minute crystals, or other objects, which are best placed on black velvet, and viewed through the prism as a whole, no slit being required.

Examples of the application of the preceding methods of observation.

253. The peculiar properties of paper washed with tincture of turmeric or stramonium seeds, of yellow uranite, and other highly sensitive substances, come out in a remarkable manner under the modes of examination described in this paper. I need not say that such is the case with solutions of sulphate of quinine, or horse-chestnut bark, or other clear and highly sensitive media, seeing that in this case the appearance due to fluorescence is obvious to common observation. If a piece of horse-chestnut bark be put to float in a glass of water close to the hole covered by the principal absorbent, the appearance of the descending streams of solution of esculine is very singular and beautiful. My present object is however rather to illustrate the power of these methods

difficulty the sensibility of white paper on a day of continuous clouds and rain. Even cotton-wool, which stands very much lower in the scale, is shown by the use of absorbing media with ordinary daylight to be sensitive. In the case of such substances as bone, ivory, white leather, the white part of a quill, which stand much higher in the scale, the most inexperienced observer could hardly fail instantly to detect the fluorescence. All plates of colourless glass which I have examined, and other pieces which were of such a shape as to admit of being looked into edgeways to a considerable depth, were found by the second combination to be sensitive. Crystals of sulphate of quinine, which may be readily prepared from the disulphate of commerce, show their fluorescence extremely well by the first combination. These crystals are much less sensitive than their solution, and the light which they emit is of a much deeper blue. It must in reality be of a very deep blue colour; for it nearly matches the fluorescent light of fluor-spar, although when the crystals are viewed under the violet glass the tint in both cases appears comparatively pale, from contrast with the violet. A solution of nitrate of uranium on the other hand has only a low degree of sensibility compared with the crystals of that salt. If a drop of the solution be placed on the porcelain tablet when the hole is covered with the deep blue copper solution, it appears comparatively dark, because much more illumination is lost by the absorption of the indigo and violet than is gained by the fluorescence of the solution. But when the tablet is viewed through the complementary absorbent, the solution is seen to be more luminous than the tablet, and to emit yellow rays, which are not found in the incident light.

The reactions of quinine mentioned in my former paper (Arts. 205–208), may very conveniently be observed by means of drops of the solution placed on the tablet; and in this way it is possible to work in a perfectly satisfactory manner with excessively minute quantities of quinine. The statement there made, that the blue colour was *destroyed* by hydrochloric acid, etc., must be understood only with reference to the mode of observation there supposed to be adopted, which was sufficient for the object

extent on the addition of hydrochloric acid to a previously alkaline solution. Still there is a broad distinction between the two classes of solutions, which is all that is required. I have since extended a good deal these results, and mean to pursue the subject further. Meanwhile I may be permitted to correct an error in Art. 205, relative to the effect of hydrocyanic acid, which was there stated to develope the blue colour. The experiment was made with the acid of commerce, containing a foreign acid, to which the effect was probably due.

Comparison of the relative advantages of different modes of observation.

254. At first sight it might have been supposed that daylight could never be more than a poor substitute for sunlight in any observations relating to fluorescence. Such, however, I consider to be by no means the case. In the first place, when sunlight is used it is made to enter a room in a definite direction; whereas in using absorbing media all the rays are employed whose directions lie within a solid angle having the object examined for vertex, and the hole for base. If we leave out the part of this solid angle which corresponds to trees or houses, the part which corresponds to sky will still be so large as to make up in a good measure for the superior brilliancy of the light of the sun. In the second place, stray light is much more perfectly excluded than when a beam of sunlight, containing rays of all kinds, is admitted into a room. When indeed the use of sunlight is combined with that of absorbing media, it is possible to detect very minute degrees of sensibility. Still for general purposes I consider the methods depending upon the use of absorbing media with ordinary daylight quite comparable with, if not equal to, those methods involving the use of sunlight which are applicable to opaque bodies; I allude especially to the method of a linear spectrum. The peculiarities in the composition of the fluorescent light, when such exist, can be made out about equally well by both methods.

But when the substance to be investigated is a solution, or

wise dazzle the eye, an amount of concentration of the rays can be brought to bear on the object, which enables the observer to detect excessively minute degrees of sensibility. Thus, when the sun's light is condensed by a rather large lens, and made to pass through a strong solution of the ammoniaco-sulphate of copper, the condensed beam of violet and invisible rays serves to detect fluorescence in almost all fluids. This, however, is no great advantage; the method is in fact too powerful; and the observer is left in doubt whether the effect perceived be due to the fluid deemed to be examined, or to some impurity which it contains in an amount otherwise perhaps inappreciable. The great advantage which sunlight observations possess in the examination of substances, which, however, is only applicable to clear media, is that they enable the observer to make out the distribution of activity in the incident spectrum. In some cases this constitutes the chief peculiarity in the mode of fluorescence of a particular substance; in other cases it enables the observer to see, as it were, independently of each other, different sensitive substances existing together in solution. Another advantage of sunlight, which applies equally to clear and to opaque media, is that it enables the observer, with the assistance of a quartz train, to make out fluorescence which does not commence till that region of the spectrum which it requires a quartz train to show. But such cases are too rare to render this a point of much consequence. Of course there are observations such as those which relate to the fixed lines of the invisible rays, or to the determination of the absorbing action of a medium with regard to invisible rays of each degree of refrangibility in particular, which imperatively require sunlight: I am speaking at present only with reference to those observations of which the object is to investigate the mode of fluorescence of a particular substance.

As to the method of observation in which a prism is combined with a principal absorbent, its chief use is to determine, in the case of the more sensitive substances, the composition of the

255. Although the description of the mode of observing by means of absorbing media has run to some little length, the reader must not suppose that the observations are at all difficult. Of course observations of all kinds become more or less difficult when they are pushed to the extreme limit of refinement of which they are susceptible. But in the case of substances which are at all highly sensitive, and this comprises almost all the more interesting instances, the observations are extremely easy. I have spoken of a darkened room, which is certainly the most convenient when it can be had. But I have no doubt that an observer who could not procure such might easily arrange for himself a darkened box, which would answer the purpose. Indeed the fluorescence of highly sensitive substances, though they be opaque, may be exhibited by means of absorbing media in broad daylight.

Platinocyanides.

256. In the Report of the Twentieth Meeting of the British Association (Edinburgh, 1850, Transactions of the Sections, p. 5), is a notice by Sir David Brewster of "The Optical Properties of the Cyanurets of Platinum and Magnesia, and of Barytes and Platinum," salts which he had received from M. Haidinger of Vienna. The notice is chiefly devoted to the properties of the reflected light; but with respect to the latter of the salts, Sir David remarks that "it possesses the property of internal dispersion, the dispersed light being a *brilliant green*, while the transmitted light is *yellow*." Although the distinction between true internal dispersion and opalescence was not at the time understood, there could be little doubt from the nature of the case that the internal dispersion mentioned by Sir David Brewster was, in fact, an instance of the former of these phenomena; but I could not try for want of a specimen of the salt. Some months ago I received from M. Haidinger a specimen of the first of the salts mentioned at the beginning of this paragraph, namely M. Quadrat's cyanide of platinum and magnesium, a salt of great optical interest on account of the remarkable metallic reflexion which it exhibits.

aspects is, i
possesses ir
some of th
it to the f
calcium, ba
salts last n
David Brew
being of dif
that the pl
mere water.
of potassiu
cases that
pernitrate
produced w
than even y
a very strik
with the sec
of potassium
look like wa
mixed, a pre
body with a
is also sensit
cyanides are
or rather cla
is attached t
manner (thou
other organic
other two case
and secondly,
consists in th
myself at pre
extended stud

nt of concentration of the rays can
object, which enables the observer
degrees of sensibility. Thus, when
by a rather large lens, and made
ution of the ammoniaco-sulphate of
of violet and invisible rays serves
ost all fluids. This, however, is no
d is in fact too powerful; and the
er the effect perceived be due to the
, or to some impurity which it con-
perhaps inappreciable. The great
ervations possess in the examination
r, is only applicable to clear media,
erver to make out the distribution
pectrum. In some cases this con-
y in the mode of fluorescence of a
er cases it enables the observer to
ly of each other, different sensitive
in solution. Another advantage
ually to clear and to opaque media,
er, with the assistance of a quartz
ence which does not commence till
n which it requires a quartz train
are too rare to render this a point
course there are observations such
e fixed lines of the invisible rays,
the absorbing action of a medium
ys of each degree of refrangibility
ely require sunlight: I am speaking
nce to those observations of which
e mode of fluorescence of a particular

rvation in which a prism is combined
ts chief use is to determine, in the
substances, the composition of the

On the Optical Properties of a recently discovered Salt of Quinine.

[From the *Report of the British Association*, Belfast, 1852, pp. 15–16.]

This salt is described by Dr Herapath in the *Philosophical Magazine* for March 1852, and is easily formed in the way there recommended, namely, by dissolving disulphate of quinine in warm acetic acid, adding a few drops of a solution of iodine in alcohol, and allowing the liquid to cool, when the salt crystallizes in thin scales reflecting (while immersed in the fluid) a green light with a metallic lustre. When taken out of the fluid the crystals are yellowish-green by reflected light, with a metallic aspect. The following observations were made with small crystals formed in this manner; and an oral account of them was given at a meeting of the Cambridge Philosophical Society, shortly after the appearance of Dr Herapath's paper.

The crystals possess in an eminent degree the property of polarizing light, so that Dr Herapath proposed to employ them instead of tourmalines, for which they would form an admirable substitute, could they be obtained in sufficient size. They appear to belong to the prismatic system; at any rate they are symmetrical (so far as relates to their optical properties and to the directions of their lateral faces) with respect to two rectangular planes perpendicular to the scales. These planes will here be called respectively the *principal plane of the length* and the *principal plane of the breadth*, the crystals being usually longest in the direction of the former plane.

principal plane of the length the crystals are transparent, and nearly colourless, at least when they are as thin as those which are usually formed by the method above mentioned. But with respect to light polarized in the principal plane of the breadth, the thicker crystals are perfectly black, the thinner ones only transmitting light, which is of a deep red colour.

When the crystals are examined by light reflected at the smallest angle with which the observation is practicable, and the reflected light is analysed, so as to retain, first, light polarized in the principal plane of the length, and secondly, light polarized in the other principal plane, it is found that in the first case the crystals have a vitreous lustre, and the reflected light is colourless; while in the second case the light is yellowish-green, and the crystals have a metallic lustre. When the plane of incidence is the principal plane of the length, and the angle of incidence is increased from 0° to 90°, the part of the reflected pencil which is polarized in the plane of incidence undergoes no remarkable change, except perhaps that the lustre becomes somewhat metallic. When the part which is polarized in a plane perpendicular to the former is examined, it is found that the crystals have no angle of polarization, the reflected light never vanishing, but only changing its colour, passing from yellowish-green, which it was at first, to a deep steel-blue, which colour it assumes at a considerable angle of incidence. When the light reflected in the principal plane of the breadth is examined in a similar manner, the pencil which is polarized in the plane of incidence undergoes no remarkable change, continuing to have the appearance of being reflected from a metal, while the other or colourless pencil vanishes at a certain angle, and afterwards reappears, so that in this plane the crystals have a polarizing angle.

If, then, for distinction's sake, we call the two pencils which the crystals, as belonging to a doubly refracting medium, transmit independently of each other, *ordinary* and *extraordinary*, the former being that which is transmitted with little loss, we may say, speaking approximately, that the medium is transparent with respect to the ordinary ray and opaque with respect to the extra-

light merely be used, both refracted pencils are produced, and the corresponding reflected pencils are viewed together; but by analysing the reflected light by means of a Nicol's prism, the reflected pencils may be viewed separately, at least when the observations are confined to the principal planes. The crystals are no doubt biaxal, and the pencils here called ordinary and extraordinary are those which in the language of theory correspond to different sheets of the wave surface. The reflecting properties of the crystals may be embraced in one view by regarding the medium as not only doubly refracting and doubly absorbing, but *doubly metallic*. The *metallicity*, so to speak, of the medium of course alters continuously with the point of the wave surface to which the pencil considered belongs, and doubtless is not mathematically null even for the ordinary ray.

If the reflexion be really of a metallic nature, it ought to produce a relative change in the phases of vibration of light polarized in and perpendicularly to the plane of incidence. This conclusion the author has verified by means of the effect produced on the rings of calcareous spar. Since the crystals were too small for individual examination in this experiment, the observation was made with a mass of scales deposited on a flat black surface, and arranged at random as regards the azimuth of their principal planes. The direction of the change is the same as in the case of a metal, and accordingly the reverse of that which is observed in total internal reflexion.

In the case of the extraordinary pencil the crystals are least opaque with respect to red light, and accordingly they are less metallic with respect to red light than to light of higher refrangibility. This is shown by the green colour of the reflected light when the crystals are immersed in fluid, so that the reflexion which they exhibit as a transparent medium is in a good measure destroyed.

The author has examined the crystals for a change of refrangibility, and found that they do not exhibit it. Safflower-red, which possesses metallic optical properties, does change the refrangibility

the latter cause is red, besides which it is totally different in other respects from regularly reflected light.

In conclusion, the author observed that the general fact of the reflexion of coloured polarized pencils had been discovered by Sir David Brewster in the case of chrysammate of potash*, and in a subsequent communication he had noticed, in the case of other crystals, the difference of effect depending upon the azimuth of the plane of incidence†. Accordingly, the object of the present communication was merely to point out the intimate connexion which exists (at least in the case of the salt of quinine) between the coloured reflexion, the double absorption, and the metallic properties of the medium.

Note added during printing.—When the above communication was made to the Association, the author was not aware of M. Haidinger's papers on the subject of the coloured reflexion exhibited by certain crystals. The general phenomenon of the reflexion of oppositely polarized coloured pencils had in fact been discovered independently by M. Haidinger and by Sir David Brewster, in the instances, respectively, of the cyanide of platinum and magnesium, and of the chrysammate of potash. A brief notice of the optical properties of the former crystal will be found in *Poggendorff's Annalen,* Bd. LXVIII. (1846), S. 302, and further communications from M. Haidinger on the subject are contained in several of the subsequent volumes of that periodical. The relation of the coloured reflexion to the azimuth of the plane of incidence was noticed by M. Haidinger from the first.

* Report of the Meeting of the British Association at Southampton, 1846, Part II. p. 7.

† *Ibid.* Edinburgh, 1847, p. 5.

On the Change of Refrangibility of Light and the Exhibition thereby of the Chemical Rays.

[From the *Proceedings of the Royal Institution of Great Britain*, I, pp. 259–264; Friday evening lecture, *Feb.* 18, 1853. Also *Pogg. Ann.* LXXXIX, 1853, pp. 627–8.]

Before proceeding to the more immediate subject of the Lecture, it was necessary to refer to certain discoveries of Sir John Herschel and Sir David Brewster, more especially as it was the discovery by the former of these philosophers of the epipolic dispersion of light, and of the peculiar analysis of light which accompanies the phenomenon, that led to the researches respecting the change of refrangibility.

When a weak acid solution of quinine is prepared, by dissolving, suppose, one part of the commercial disulphate in 200 parts of water acidulated with sulphuric acid, a fluid is obtained which appears colourless and transparent when viewed by transmitted light, but which exhibits nevertheless in certain aspects a peculiar sky-blue colour. This colour of course had frequently been noticed; but it is to Sir John Herschel that we owe the first analysis of the phenomenon*. He found that the blue light emanates in all directions from a very thin stratum of fluid adjacent to the surface (whether it be the free surface or the surface of contact of the fluid with the containing glass vessel), by which the incident rays enter the fluid. His experiments clearly show that what here takes place is not a mere *subdivision* of light into a portion which is dispersed and a portion which passes on, but an actual *analysis*. For after the rays have once passed through the stratum from which the blue dispersed light comes, they are

the transmitted light had undergone, the further nature of which did not at the time appear, Sir John Herschel made use of the term "epipolized."

Sir David Brewster had several years before discovered a remarkable phenomenon in an alcoholic solution of the green colouring matter of leaves, or, as it is called by chemists, chlorophyll. This fluid when of moderate strength and viewed across a moderate thickness is of a fine emerald green colour; but Sir David Brewster found that when a bright pencil of rays, formed by condensing the sun's light by a lens, was admitted into the fluid, the path of the rays was marked by a *bright beam* of a *blood red colour**. This singular phenomenon he has designated *internal dispersion*. He supposed it to be due to suspended particles which reflected a red light, and conceived that it might be imitated by a fluid holding in suspension an excessively fine coloured precipitate. A similar phenomenon was observed by him in a great many other solutions, and in some solids; and in a paper read before the Royal Society of Edinburgh in 1846 he has entered fully into the subject†. In consequence of Sir John Herschel's papers, which had just appeared, he was led to examine a solution of sulphate of quinine; and he concluded from his observations that the "epipolic" dispersion of light exhibited by this fluid was only a particular instance of internal dispersion, distinguished by the extraordinary rapidity with which the rays capable of dispersion were dispersed.

The Lecturer stated, that, having had his attention called some time ago to Sir John Herschel's papers, he had no sooner repeated some of the experiments than he felt an extreme interest in the phenomenon. The reality of the epipolic analysis of light was at once evident from the experiments; and he felt confident that certain theoretical views respecting the nature of light had only to be followed fearlessly into their legitimate consequences, in order to explain the real nature of epipolized light.

The exhibition of a richly coloured beam of light in a perfectly

by the solutions of quinine and chlorophyll as one and the same phenomenon. The latter fluid, as has been already stated, disperses light of a blood red colour. When the transmitted light is subjected to prismatic analysis, there is found a remarkably intense band of absorption in the red, besides certain other absorption bands, of less intensity, in other parts of the spectrum. Nothing at first seemed more likely than that, in consequence of some action of the ultimate molecules of the medium, the incident rays belonging to the absorption band in the red, withdrawn, as they certainly were, from the incident beam, were given out in all directions, instead of being absorbed in the manner usual in coloured media. It might be supposed that the incident vibrations of the luminiferous ether generated synchronous vibrations in the ultimate molecules, and were thereby exhausted, and that the molecules in turn became centres of disturbance to the ether. The general analogy between the phenomena exhibited by the solutions of chlorophyll and of quinine would lead to the expectation of absorption bands in the light transmitted by the latter. If these bands were but narrow, the light belonging to them might not be missed in the transmitted beam, unless it were specially looked for; and the beam might be thus "epipolized," without, to ordinary inspection, being changed in its properties in any other respect. But on subjecting the light to prismatic analysis, first with the naked eye, and then with a magnifying power, no absorption bands were perceived.

A little further reflection showed that even the supposition of the existence of these bands would not alone account for the phenomenon. For the rays producing the dispersed light (if we confine our attention to the thin stratum in which the main part of the dispersion takes place) are exhausted by the time the incident light has traversed a stratum the fiftieth of an inch thick, or thereabouts, whereas the dispersed rays traverse the fluid with perfect freedom. This indicates a *difference of nature* between the blue-producing rays and the blue rays produced. Now, as the Lecturer stated, he felt very great confidence in the principle that the nature of light is

At first he took for granted that there could be no change of refrangibility. The refrangibility of light had hitherto been regarded as an attribute absolutely invariable*. To suppose that it had changed would, on the undulatory theory, be equivalent to supposing that periodic vibrations of one period could give rise to periodic vibrations of a different period, a supposition presenting no small mechanical difficulty. But the hypotheses which he was *obliged* to form on adopting the other alternative, namely, that the difference of nature had to do with the state of polarization, were so artificial as to constitute a theory which appeared utterly extravagant. He was thus led to contemplate the possibility of a change of refrangibility. No sooner had he dwelt in his mind on this supposition, than the mystery respecting the nature of epipolized light vanished; all the parts of the phenomenon fell naturally into their places. So simple did the whole explanation become, when once the fundamental hypothesis was admitted, that he could not help feeling strongly impressed that it would turn out to be true. Its truth or fallacy was a question easily to be decided by experiment; the experiments were performed, and resulted in its complete establishment.

The Lecturer then described what may be regarded as the fundamental experiment. A beam of sunlight was reflected horizontally through a vertical slit into a darkened room, and a pure spectrum was formed in the usual manner, namely, by transmitting the light through a prism at the distance of several feet from the slit, and then through a lens close to the prism. In the actual experiment, two or three prisms were used, to produce a greater angular separation of the colours. Instead of a screen, there was placed at the focus of the lens a vessel containing a solution of sulphate of quinine. It was found that the red, orange, etc., in fact, nearly the whole of the visible rays, passed through the fluid as if it had been mere water. But on arriving about the middle of the violet, the path of the rays within the fluid was marked

* It is true that the phenomenon of phosphorescence is in a certain sense an exception; but the effect is in this case a work of time which seems at once to

by a sky-blue light, which emanated in all directions from the fluid, as if the medium had been self-luminous. This blue light continued throughout the region of the violet, and far beyond, in the region of the invisible rays. The posterior surface of the luminous portion of the fluid marked the distance to which the incident rays were able to penetrate into the medium before they were exhausted. This distance, which at first exceeded the diameter of the vessel, decreased with great rapidity, so that in the greater part of the invisible region it amounted to only a very small fraction of an inch. The fixed lines of the extreme violet, and of the more refrangible invisible rays, were exhibited by dark planes interrupting the dispersed light. When a small portion of the incident spectrum was isolated, by stopping the rest by a screen, and the corresponding beam of blue dispersed light was refracted sideways by a prism held to the eye, it was found to consist of light having various degrees of refrangibility, with colour corresponding, the more refrangible rays being more abundant than the less refrangible. The nature of epipolized light is now evident; it is nothing but light from which the highly refrangible invisible rays have been withdrawn by transmitting it through a solution of quinine, and does not differ from light from which those rays have been withdrawn by any other means.

The fundamental experiment, excepting that part of it which relates to the analysis of the dispersed light, was then exhibited by means of the powerful voltaic battery belonging to the Institution, which was applied to the combustion of metals. The rays emanating from the voltaic arc were applied to form a pure spectrum, which was received on a slab of glass coloured by peroxide of uranium, a medium which possesses properties similar to those of a solution of sulphate of quinine in a still more eminent degree.

The difference of nature of the illumination produced by a change of refrangibility, or "true internal dispersion," from that due to the mere scattering of light, may be shown in a very instructive form by placing paper washed with sulphate of quinine, or a screen of similar properties, so as to receive a long

of course to be thrown obliquely upwards; whereas it is actually decomposed by the prism into two bands, one ascending obliquely, and consisting of the usual colours of the spectrum in their natural order, the other running horizontally, and extending far beyond the more refrangible end of the former. Whatever be the screen, the horizontal band is always situated below the oblique, since there appears to be no exception to the law, that when the refrangibility of light is changed in this manner it is *always lowered.*

The general appearance of some highly "sensitive" media in the invisible rays was then exhibited by means of the flame of sulphur burning in oxygen, a source of these rays which Dr Faraday, to whose valuable assistance the Lecturer was much indebted, had in some preliminary trials found very efficacious. The chief media used were articles made of glass coloured by uranium, and solutions of quinine, of horse-chestnut bark, and of the seeds of the datura stramonium. A tall cylindrical jar filled with water showed nothing remarkable; but when a solution of horse-chestnut bark was poured in, the descending fluid was strongly luminous. The experiment was varied by means of white paper on which words had been written with a pretty strong solution of sulphate of quinine, an alcoholic solution of the seeds of the datura stramonium, and a purified aqueous solution of horse-chestnut bark. By gas-light the letters were invisible; but by the sulphur light, especially when it had been transmitted through a blue glass, which transmits a much larger proportion of the invisible than of the visible rays, the letters appeared luminous, on a comparatively dark ground. A glass vessel containing a thin sheet of a very weak solution of chromate of potash allowed the letters to be seen as well, or very nearly as well as before, when it was interposed between the eye and the paper; but when it was interposed between the flame and the paper the letters wholly disappeared,—the medium being opaque with respect to the rays which caused the letters to be luminous, but transparent with respect to the rays which they emitted.

to the visible rays. It is sufficient to interpose the medium in the path of the incident rays, and to notice the effect. Again, the effect of various flames and other sources of light on solutions of quinine, and on similar media, indicates the richness or poverty of those sources with respect to the highly refrangible invisible rays. Thus, the flames of alcohol, of hydrogen, etc., of which the illuminating power is so feeble, were found to be very rich in invisible rays. This was still more the case with a small electric spark, while the spark from a Leyden jar was found to abound in rays of excessively high refrangibility. These highly refrangible rays were stopped by glass, but passed freely through quartz. These results, and others leading to the same conclusion, had induced the Lecturer to order a complete train of quartz. A considerable portion of this was finished before the end of last August, and was applied to the examination of the solar spectrum. A spectrum was then obtained extending beyond the visible spectrum, that is, beyond the extreme violet, to a distance at least double that of the formerly known chemical spectrum. This new region was filled with fixed lines like the regions previously known.

But a spectrum far surpassing this was obtained with the powerful electrical apparatus belonging to the Institution. The voltaic arc from metallic points furnished a spectrum no less than *six or eight times* as long as the visible spectrum. This was in fact the spectrum which had already been exhibited in connexion with the fundamental experiment. The prisms and lens which the Lecturer had been employing in forming the spectrum were actually made of quartz. The spectrum thus obtained was filled from end to end with bright bands. When a piece of glass was interposed in the path of the incident rays, the length of the spectrum was reduced to a small fraction of what it had been, all the more refracted part being cut away. A strong discharge of a Leyden jar had been found to give a spectrum at least as long as the former, but not, like it, consisting of nothing but isolated bright bands.

He stated also that during the winter, even when the sun shone clearly, it was not possible to see so far as before. As spring advanced he found the light continually improving, but still he was not able to see so far as he had seen at the end of August. It was plain that the earth's atmosphere was by no means transparent with respect to the most refrangible of the rays belonging to the solar spectrum.

In conclusion, there was exhibited the effect of the invisible rays coming from a succession of sparks from the prime conductor of a large electrifying machine, in illuminating a slab of glass coloured by uranium.

On the Cause of the Occurrence of Abnormal Figures in Photographic Impressions of Polarized Rings.

[From the *Philosophical Magazine*, VI, *Aug.* 1853, pp. 107—113.
Also *Pogg. Ann.* XC, 1853, pp. 488—497.]

The object of the following paper is to consider the theory of some remarkable results obtained by Mr Crookes in applying photography to the study of certain phenomena of polarization. An account of these results, taken from the *Journal of the Photographic Society*, is published in the last number of the *Philosophical Magazine**.

In the ordinary applications of photography, certain objects and parts of objects are to be represented which differ from one another in colour, or in brightness, or in both, according to the nature of the substances, and the way in which the lights and shadows fall. In the photograph the objects are represented as simply light or dark. Inasmuch as the photographic power, in relation to a given sensitive substance, of a heterogeneous pencil of rays is not proportional to its illuminating power, the darkness of the objects in a negative photograph is not proportional to, nor even always in the same order of sequence as, their brightness as they appear to the eye. Still, the outlines of the objects and of their parts are faithfully preserved. For although it is conceivable that two adjacent parts of an object, which the eye instantly distinguishes by their colour, should reflect rays of almost exactly equal photographic power in relation to the particular sensitive substance employed, so as to be absolutely undistinguishable on the photograph, or on the other hand, that

the forms of a set of objects, suppose coloured patterns, or a painting of the rings of crystals, should be changed in this way by the substitution of one set of outlines for another are so very peculiar, that the chances may be regarded as infinity to one that no such changes of form will be produced to any material extent.

But when photography is applied phenomena of interference, such as the means of polarized light, the case is different. To take a particular in rings of calcareous spar to be viewe of polarization of the polarizer and a to give the black cross; and consider place in going outwards from the ce a direction inclined at angles of 45° to At first there are evident alternations of intensity, soon the eye, which under such circumstances is but a bad judge of differences of intensity even when the lights to be compared have the same colour, can no longer perceive the differences of illumination, but judges entirely by the difference of tint. The same takes place with nitre, sugar, and other colourless biaxal crystals. Except in the immediate neighbourhood of the optic axis or axes, the rings, which owe their existence and their forms in the first instance to the laws of double refraction and of the interference of polarized light, are in other respects created and their forms determined by the condition of maximum contrast of tint.

Now consider what takes place when an image of such a system is thrown on a sensitive plate, prepared suppose by means of bromide of silver. The rays of any one refrangibility would together form a regular system of rings, which, if these rays were alone present, and if the refrangibility were comprised within the limits between which the substance is acted on, would impress on the plate a system of rings exactly like those seen by means of the same homogeneous rays, provided they belong to

certain alternations of light and shade corresponding to alternations in the *total photographic intensity* of the rays which had acted on the plate, without any distinction being preserved between the action of rays of one refrangibility and that of rays of another; whereas, when the rings are viewed directly, the eye catches the differences of tint without noticing the difference of intensity, except in the neighbourhood of the optic axis or axes. Of course I am now speaking only of the alternations perceived in following a line drawn across the rings, not of the dark brushes, or of the variation of intensity perceived in passing along a given ring. Hence, when heterogeneous light is used, the circumstances which determine the rings are so different in the two cases that it is no wonder that the character of the rings seen on a photograph should differ in some respects from that of the rings seen directly.

But not only is a difference of character indicated as likely to take place; a more detailed consideration of the actual mode of superposition will serve to explain some of the leading features of the abnormal rings as observed by Mr Crookes. Let us take for example calcareous spar, and suppose the transmitted rays to be all of the same refrangibility. In this case the intensity along a given radius vector, drawn from the centre of the cross, varies as the square of the sine of half the retardation of phase of the ordinary relatively to the extraordinary pencil (see Airy's Tract). If i be the angle of incidence, the retardation varies nearly as $\sin^2 i$; and if $\sin^2 i = r$, we may take, as representing the variations of intensity I,

$$I = \sin^2(mr^2) = \tfrac{1}{2}(1 - \cos 2mr^2) \quad \ldots\ldots\ldots\ldots\ldots (1).$$

In this expression m is a constant depending upon the refrangibility of the rays. In the case of calcareous spar the tints of the rings follow Newton's scale, and m is very nearly proportional to the reciprocal of the wave-length.

Suppose now that rays of two different degrees of refran-

producing separately. The latter supposition, if not strictly true, will no doubt be approximately true if the plate be not too long exposed. Then, if m' be what the parameter m becomes for the second system, we may represent the variation of intensity along a given radius vector by

$$I = \sin^2(mr^2) + \sin^2(m'r^2) = 1 - \cos(\overline{m - m'}r^2)\cos(\overline{m + m'}r^2) \ldots (2).$$

Suppose the refrangibilities of the two systems to be moderately different; then the difference between the two parameters m, m' will be small, but not extremely small, compared with either of them. Hence of the two factors in the expression for I the second will fluctuate a good deal more rapidly than the first, and will be that which mainly determines the radii, etc. of the rings. If the first factor were constant and equal to 1, its value when $r = 0$, the expression (2) would be of precisely the same form as (1), the parameter being the mean of the two, m, m'. However, the first factor is not constant, but decreases as r increases, and presently vanishes, and then changes sign. Hence the rings become less distinct than with homogeneous rays, and presently there takes place a sort of dislocation amounting to half an order, that is, the bright rings beyond a certain point, or in other words, outside the circle determined by a certain value of r, correspond, in regard to the series formed by their radii, with the dark rings inside this circle, and *vice versâ*. At some distance beyond that at which the dislocation takes place the rings become very distinct again; but it is useless to trace further the variations of the expression for I, because the circumstances supposed to exist in forming that expression are too remote from those of actual experiment to allow the interpretation of the formula to be carried far.

According to the numerical values of m and m', a dark ring might be converted, by the change of sign of the factor $\cos(m - m')r^2$, into a bright ring, or a bright ring into a dark ring, or a ring of either kind might be rendered broader or narrower than it would regularly have been. The coalescence of the fourth and fifth bright rings in Mr Crookes's photographs when bromide of

should act, nor even that the curve of photographic intensity should admit of two distinct maxima within the spectrum. Suppose that rays of all refrangibilities lying within certain limits pass through the crystal and fall on the plate. For the sake of obtaining an expression which admits of being worked out numerically without too much trouble, and yet results from a hypothesis not very remote from the circumstances of actual experiment, I will suppose the total photographic power of the rays whose parameters lie between m and $m+dm$ to be proportional to $\sin m\,dm$ between the limits $m=2\pi$ and $m=3\pi$, and to vanish beyond those limits. Since m is very nearly proportional to the reciprocal of the wave-length, and the ratio 3π to 2π or 3 to 2 is nearly that of the wave-lengths of the fixed lines D, H, this assumption corresponds to the supposition that the rays less refrangible than D are inefficient, that the action there commences, then increases according to a certain law, attains a maximum, decreases, and finally vanishes at H. The action would really terminate at H if a bath of a solution of sulphate of quinine of a certain strength were used. On this assumption, and supposing, as before, that the rays of different refrangibilities act independently of each other, we have

$$I=\int_{2\pi}^{3\pi}\sin^2(mr^2)\sin m\,dm.$$

On working out this expression, and writing x for $2r^2$, we find

$$I=1+\frac{\cos(\frac{1}{2}\pi x)\cos(\frac{5}{2}\pi x)}{x^2-1}\quad\text{.....(3).}$$

As the full discussion of this formula presents no difficulty it may be left to the reader. The last factor in the numerator of the fraction is that where fluctuations correspond to the rings. Whenever x passes through an odd integer greater than 1 the first factor changes sign, and there is a dislocation or displacement of half an order, but when x passes through the value 1 the denominator changes sign along with both factors of the numerator, and there is no dislocation. When x becomes con-

of rings. In passing from one ring to its consecutive the angle $\frac{5}{2}\pi x$ increases by 2π, and therefore x by 0·8. The sixteenth part of this, or 0·05, is the increment of x in the table. Each vertical column corresponds to one order. The value of x corresponding to any number in the table will be found by adding together the numbers in the top and left-hand columns.

x	0·00	0·80	1·60	2·40	3·20	4·00	4·80
·00	0·000	0·142	0·481	0·830	1·033	1·067	1·014
·05	0·080	0·223	0·543	0·860	1·037	1·060	1·010
·10	0·295	0·418	0·667	0·905	1·032	1·044	1·005
·15	0·619	0·692	0·829	0·955	1·020	1·021	1·001
·20	1·000	1·000	1·000	1·000	1·000	1·000	1·000
·25	1·377	1·293	1·154	1·033	0·977	0·979	1·001
·30	1·692	1·527	1·268	1·051	0·956	0·964	1·004
·35	1·898	1·669	1·327	1·054	0·939	0·956	1·008
·40	1·963	1·702	1·333	1·045	0·932	0·956	1·012
·45	1·881	1·791	1·286	1·030	0·936	0·963	1·013
·50	1·669	1·465	1·205	1·014	0·950	0·974	1·012
·55	1·356	1·243	1·103	1·004	0·973	0·987	1·007
·60	1·000	1·000	1·000	1·000	1·000	1·000	1·000
·65	0·654	0·775	0·913	1·004	1·027	1·010	0·991
·70	0·373	0·600	0·853	1·013	1·049	1·015	0·983
·75	0·192	0·499	0·826	1·024	1·063	1·016	0·977

A curve of intensity might easily be constructed from this table by taking ordinates proportional to the numbers in the table, and abscissæ proportional to the values of r, and therefore to the square roots of the numbers 0, 1, 2, 3, 4, etc. But the form of the curve will be understood well enough either from the formula (3), or from an inspection of the numbers in the table.

It will be seen that in the first three columns the numbers lying between the horizontal lines beginning ·20 and ·60 correspond to bright rings, and the remainder of each column, together with the beginning of the next, corresponds to a dark ring. But the

nly down to its mean value unity. A similar
curs in the seventh column, but here the whole
ntensity is comparatively small.

se of calcareous spar the character of the rings is
the way round, but in the photographs of the rings
ew feature presents itself. Mr Crookes's figure of
l rings of nitre is rather too small to be clear, but
ssistance of his description there is no difficulty in
hat takes place. With reference to these photographs
"But here a remarkable dislocation presented itself;
of the interior rings, instead of retaining its usual
appeared as if broken in half, the halves being
sed and depressed towards the neighbouring rings."

This effect admits of easy explanation as a result of the superposition of systems of rings which separately are perfectly regular, when we consider that the poles of the lemniscates of the several elementary systems do not coincide, since in nitre the angle between the optic axes increases from the red to the blue. Now the change of character which may be described as a displacement of half an order is due to the circumstance that the smaller rings corresponding to the more refrangible rays are, as it were, overtaken by the larger rings corresponding to the less refrangible. It is plain that the variation of position of the poles of the lemniscates would tend to retard this effect in directions lying outside the optic axes, and to accelerate it in directions lying between those axes. Hence what was a bright ring in one part of its course would become a dark ring in another part, so that each quadrant would exhibit a dislocation of half an order in the rings. In order to show this dislocation to the greatest advantage, a crystal of a certain thickness should be used. With a very thin crystal there would be no dislocation of this nature, but only a displacement like that which takes place with calcareous spar. With a very thick crystal the effect of the chromatic variation of position of the optic axes would be too much exaggerated.

whose existence is purely hypothetical, such for example as invisible rays accompanying, but distinct from, visible rays of the same refrangibility. Some of the minor details of the abnormal rings may require further explanation or more precise calculation; but such calculations are of no particular interest unless the phenomena offered grounds for suspecting the agency of hitherto unrecognized causes.

The difference between the photographs taken with iodide and bromide of silver is easily explained, when we consider the manner in which those substances are respectively affected by the rays of the spectrum. With iodide of silver there is such a concentration of photographic power extending from about the fixed line G of Fraunhofer to a little beyond H, that even when white light is employed we may approximately consider that we are dealing with homogeneous rays. On this account, and not because the rays of high refrangibility are capable of producing a more extended system of rings than those of low refrangibility, the rings visible on the photograph are much more numerous than those seen directly by the eye with the same white light. Moreover, the rings do not exhibit the same abnormal character as with bromide of silver, in relation to which substance the photographic power of the rays is more diffused over the spectrum.

It is not possible to place the eye and a sensitive plate prepared with bromide of silver under the same circumstances with regard to the formation of abnormal rings. It would be easy, theoretically at least, to place the eye and the plate in the same circumstances as regards rings, by using homogeneous light; but then, I feel no doubt, the rings visible on the plate would be as regular as those seen by the eye. On the other hand, if differences of colour exist in the figure viewed by the eye, they inevitably arrest the attention, and it is impossible to get rid of them without at the same time rendering the light so nearly homogeneous that on that account nothing abnormal would be shown. Hence Mr. Crookes's abnormal rings afford a very curious

On the Metallic Reflexion exhibited by Certain Non-Metallic Substances.

[From the *Philosophical Magazine*, VI, *Dec.* 1853, pp. 393—403. Also *Pogg. Ann.*, XCI, 1854, pp. 300—14; *Ann. de Chimie*, XLVI, 1856, pp. 504—8.]

In the October Number of the *Philosophical Magazine* is a translation of a paper by M. Haidinger of Vienna, containing an account of his observations relating to the optical properties of Herapathite. In this paper he refers to a communication which I made to the British Association at the meeting at Belfast*; and indeed one great object of his examination of this salt was to see whether a law which he had discovered, and already extensively verified, relating to the connexion between the reflected and transmitted tints of bodies which have the property of reflecting a different tint from that which they transmit, would be verified in this case. The report of my communication published in the Abbé Moigno's *Cosmos*† had led him to suppose that my observations were at variance with his law.

My attention was first directed to this subject while engaged in some observations on safflower-red (carthamine), which I was led to examine with reference to its fluorescence. In following out the connexion which I had observed to exist between the absorbing power of a medium and its fluorescence, I was induced to notice particularly the composition of the light transmitted by the powder; and I found that the medium, while it acted powerfully on all the more refrangible rays of the visible spectrum, absorbed green light with remarkable energy. I need not now describe the mode of absorption more particularly. During these experiments I was struck with the metallic yellowish-green reflexion which this substance exhibits. It occurred to me that the almost metallic opacity of the

was connected with the reflexion of a greenish light with a metallic aspect. I found, in fact, that the medium agreed with a metal in causing a retardation in the phase of vibration of light polarized perpendicularly to the plane of incidence relatively to light polarized in that plane. The observation was made by reflecting light polarized at an azimuth of about 45° from the surface of the medium to be examined, the angle of incidence being considerable, about equal to the angle of maximum polarization, and viewing the reflected light through a Nicol's prism capped by a plate of calcareous spar cut perpendicularly to the axis. Now by using different absorbing media in succession, it was found that with red light, for which safflower-red is comparatively transparent, the reflected light was sensibly plane-polarized, while for green and blue light the ellipticity was very considerable.

In the case of a transparent medium, light would be polarized by reflexion, or at least very nearly so, at a proper angle of incidence. Hence if the light reflected by such a medium as safflower-red were decomposed into two pencils, one, which for distinction's sake may be called the ordinary, polarized in the plane of incidence, and the other, or extraordinary, polarized perpendicularly to the plane of incidence, the extraordinary pencil would vanish at the polarizing angle, except in so far as the laws of the reflexion deviate from those belonging to a transparent substance. Hence the light remaining in the extraordinary pencil might be expected to be more distinctly related to the light absorbed with such energy. Accordingly, it was found that under these circumstances the extraordinary pencil (in the case of safflower-red) was of a very rich green colour, whereas without analysis the light was of a yellowish-green colour. Similar observations were extended to specular iron.

These phenomena recalled to my mind a communication which Sir David Brewster made at the meeting of the British Association at Southampton in 1846*; and on referring to his paper, I found that the appearance of differently coloured ordinary and extraordinary pencils in the light reflected by safflower-red was the same phenomenon as he has there described with reference to

The observations above-mentioned were made towards the end of 1851. Accordingly, when Dr Herapath's first paper on the new salt of quinine appeared*, I was prepared to connect the en reflected light with an intense absorbing action , to green rays. Having prepared some crystals) his directions, I was readily able to trace the pro- e absorption in the case of light polarized in a plane lar to what is usually the longest dimension of the plates, and to observe how the light passed from red s the thickness increased. Even the thickest of these as so thin as to show hardly any colour by light he plane of the length. The result of crossing two ; of course obvious to any one who is conversant optics. The intense absorption was readily con- he metallic reflexion. An oral account of these ns was given at a meeting of the Cambridge Philo- l Society on March 15, 1852; but it was not till the obser- vations were a second time described, with some slight additions, at the meeting of the British Association at Belfast, that any account of them was published. A notice of this communication appeared in the *Athenæum* of September 25, 1852, and from this the report in *Cosmos* seems to be taken, though the latter is not free from errors. In the report in the *Athenæum*, the colour of the more rapidly absorbed pencil is briefly described in these words: "But with respect to light polarized in the principal plane of the breadth, the thicker crystals are perfectly black, the thinner ones only transmitting light, which is of a deep red colour." The comparative transparency of the crystals with regard to red light is afterwards expressly connected with the green colour of the light reflected as if by a metal. But in the report in *Cosmos*, the passage just quoted is replaced by "tandis que pour le cas de la lumière polarisée dans le plan principal de la largeur ils sont opaques et noirs, quelque minces qu'ils soient d'ailleurs." This error led M. Haidinger to suppose that my observations were opposed to his law; whereas the fact is, that, without knowing at the time what he had done, I had been led independently to a similar conclusion.

the priority of those to whom priority belongs. M. Haidinger had several years before discovered the phenomenon of the reflexion of differently coloured oppositely polarized pencils, which Sir David Brewster shortly afterwards, and independently, discovered in the case of chrysammate of potash. M. Haidinger had from the first observed a most important character of the phenomenon in the case of many crystals, namely, the orientation of the polarization of the reflected light, which Sir David Brewster does not seem to have noticed in the case of chrysammate of potash, and which perhaps was not very evident in that salt. In a later paper M. Haidinger had announced the complementary relation of the reflected and transmitted tints*. There is nothing new in employing the rings of calcareous spar as a means of detecting elliptic polarization; and the property of producing elliptic polarization in reflecting plane-polarized light had previously been observed in substances even of vegetable origin†. I am not aware, however, that the chromatic variations of the change of phase had been experimentally connected with the chromatic variations of an intense absorbing action on the part of the medium. I have hitherto mentioned but one instance of this connexion, but I shall presently have occasion to allude to another.

* In a paper of M. Haidinger's, entitled "Über den Zusammenhang der Körperfarben, oder des farbig durchgelassenen, und der Oberflächenfarben, oder des zurückgeworfenen Lichtes gewisser Körper," from the January Number of the Proceedings of the Mathematical and Physical Class of the Academy of Sciences at Vienna for the year 1852, will be found a list of M. Haidinger's previous papers on this subject. This paper contains a methodized account of the properties, with reference to surface and substance colour, of the substances up to that time examined by the author, amounting in number to thirty. For a copy of this, as well as several others of his papers, I am indebted to the kindness of the author.

† More than twenty years ago Sir David Brewster, in his well-known paper "On the Phenomena and Laws of Elliptic Polarization, as exhibited in the action of Metals upon Light," pointed out the modification produced on the rings of calcareous spar as a character of polarized light after reflexion from a metal. (*Phil. Trans.* for 1830, p. 291.) In a communication to the British Association at the meeting at Southampton in 1846, Mr Dale mentions indigo among a set of substances in which he had detected elliptic polarization by means of the rings of calcareous spar. In this case, however, he connects the property, not with the intense absorbing power of the substance, but with its high refractive index.

I do not here mention the minute degrees of ellipticity which have been detected

I think it but justice to myself to point out the error in *Cosmos* (from whence M. Haidinger derived his information respecting my observations), in consequence of which I would appear to have been guilty of a grievous oversight in the examination of Herapathite: but I would hardly have ventured to mention my observations on carthamine, etc., were it not that, when different persons arrive independently at a similar conclusion, it frequently happens that views present themselves to the mind of one which may not have occurred to another. In the present case, in stating in detail my own views as to the nature of the phenomenon, I hope to be able to add something to what has been already done by M. Haidinger and Sir David Brewster, and it seemed not out of place to mention the observations in which those views originated.

It appears, then, that certain substances, many of them of vegetable origin, have the property of reflecting (not scattering) light which is coloured and has a metallic aspect. The substances here referred to are observed to possess an exceedingly intense absorbing action with respect to rays of the refrangibilities of these which constitute the light thus reflected, so that for these rays the opacity of the substances is comparable with that of metals. Contrary, however, to what takes place in the case of metals, this intense absorbing action does not usually extend to all the colours of the spectrum, but is subject to chromatic variations, in some cases very rapid. The aspect of the reflected light, which itself alone would form but an uncertain indication, is not the only nor the principal character which distinguishes these substances. In the case of transparent substances, or those of which the absorbing power is not extremely intense (for example, coloured glasses, solutions, etc.), the reflected light vanishes, or almost entirely vanishes, at a certain angle of incidence, when it is analysed so as to retain only light polarized perpendicularly to the plane of incidence*, which is not the case with metals. In the case of the substances at present considered, the reflected light does not vanish, but at a considerable angle of incidence

* I do not here take into account the peculiar phenomena which have been

the pencil polarized perpendicularly to the plane of incidence becomes usually of a richer colour, in consequence of the removal, in a great measure, of that portion of the reflected light which is independent of the metallic properties of the medium; it commonly becomes, also, more strictly related to that light which is absorbed with such great intensity. The reflexion from a transparent medium is weakened or destroyed by bringing the medium into optical contact with another having nearly or exactly the same refractive index. Accordingly, in the case of these optically metallic substances, the colours which they reflect by virtue of their metallicity* are brought out by putting the medium in optical contact with glass or water. A remarkable character of metallic reflexion consists in the circumstance, that as the angle of incidence increases from 0°, the phase of vibration of light polarized in the plane of incidence is accelerated relatively to that of light polarized in the perpendicular plane. Accordingly, the same change takes place, with the same sign, in the case of these optically metallic substances; but the amount of the change is subject to most material chromatic variations, being considerable for those colours which are absorbed with great energy, but insensible for those colours for which the medium is comparatively transparent, so that the absorption may be neglected which is produced by a stratum of the medium having a thickness amounting to a small multiple of the length of a wave of light. If the medium be crystallized, it may happen that one only of the oppositely polarized pencils which it transmits suffers, with respect to certain colours, an exceedingly intense absorption; or, if that is the case with both pencils, that the colours so absorbed are different. It may happen, likewise, that the absorption varies with the direction of the ray within the crystal. In such cases the light reflected by virtue of the metallicity of the medium will be subject to corresponding variations, so that the medium is to be regarded as not only doubly refracting and doubly absorbing, but doubly metallic.

The views which I have just explained are derived from a combination of certain theoretical notions with some experiments.

They have need of being much more extensively verified by experiment; but, so far as I at present know, they are in conformity with observation.

To illustrate the effect of bringing a transparent medium into optical contact with an optically metallic substance, I may refer to safflower-red. If a portion of this substance be deposited on glass by means of water, and the water be allowed to evaporate, a film is obtained which reflects on the upper surface a yellowish-ight, but on the surface of contact with the glass a very een inclining to blue. A green of the latter tint appears be more truly related to the colours absorbed with greatest nergy. Similar remarks apply to the light reflected by Herapathite, according as the crystals are in air or in the mother-liquor. If a small portion of Quadratite (platinocyanide of magnesium) be dissolved on glass in a drop or two of water, and the fluid be allowed to evaporate, the tints reflected by the upper and under surfaces of the film of crystals are related to one another much in the same way as in the case of safflower-red. For a fine specimen of the salt last mentioned I am indebted to the kindness of M. Haidinger. I may mention in passing, that the platinocyanides as a class are of extreme optical interest. The crystals are generally at the same time doubly refracting, doubly absorbing, doubly metallic, and doubly fluorescent. By the last expression I mean that the fluorescence, which the crystals generally exhibit in an eminent degree, is related to directions fixed relatively to the crystal, and to the azimuths of the planes of polarization of the incident and emitted rays.

M. Haidinger has expressed the relation between the surface and substance colours of bodies by saying that they are complementary. This expression was probably not intended to be rigorously exact; and that it cannot be so, is shown by the following simple consideration. The tint of the light transmitted across a stratum of a given substance almost always, if not always, varies more or less according to the thickness of the stratum. Now one and the same tint, namely, that of the

us from regarding the reflected and transmitted tints in a general sense as complementary. But as media exist (for example, salts of sesquioxide of chromium, solutions of chlorophyll) which change their tint in a remarkable manner according to the thickness of the stratum through which the light has to pass, it is probable that instances may yet be observed in which M. Haidinger's law would appear at first sight to be violated, although in reality, when understood in the proper sense, it would be found to be obeyed. As the existence of surface-colour seems necessarily to imply a very intense absorption of those rays which are reflected according to the laws which belong to metals, it follows that it is in the very thinnest crystals or films of those which it is commonly practically possible to procure, that the transmitted tint is to be sought which is most properly complementary to the tint of the reflected light.

I will here mention another instance of the connexion between metallic reflexion and intense absorption. I choose this instance because a different explanation from that which I am about to offer has been given of a certain phenomenon observed in the substance. The instance I allude to is specular iron. As it is already known that various metallic oxides and sulphurets possess the optical properties of metals, there is nothing new in bringing forward this particular mineral as a substance of that kind. It is to the chromatic variation of the metallicity that I wish to direct attention. If light polarized at an azimuth of about 45° be reflected from a scale of this substance at about the polarizing angle, and the reflected light be viewed through a plate of calcareous spar and a Nicol's prism, it will be found, by using different absorbing media in succession, that the change of phase, as indicated by the character of the rings, while it is very evident for red light, becomes much more considerable in the highly refrangible colours. Now specular iron is almost opaque for light of all colours, but as it gives a red streak it appears that the substance-colour is red; and, in fact, it is known that very thin laminæ are blood-red by transmitted light. Accordingly, the chromatic variation of the change of phase corresponds to

angle of incidence which gives the nearest approach to complete polarization, a quantity of blue light is observed to remain. This has been explained by comparing specular iron to a substance of high dispersive power, so that the polarizing angle for red light is considerably less than for blue; and accordingly on increasing the angle of incidence, the light (which is here supposed to be analysed so as to retain only the portion polarized perpendicularly to the plane of incidence), while it becomes much less copious near the polarizing angle, becomes at the same time of a decided blue colour*. I believe, however, that the blue light is mainly due to the chromatic variation of the metallicity, the medium, considered optically, being much more metallic for blue light than for red, though it may in some measure be due to the cause previously assigned.

Specular iron is a good example of a substance forming a connecting link between the true metals and substances like safflower-red. It resembles metals in the circumstance that the absorbing power, as inferred from the chromatic variation of the metallicity, and as indicated by the tint of the streak, is not subject to the same extensive chromatic variations as in the case of colouring matters like safflower-red. It resembles safflower-red in being sufficiently transparent with respect to a portion of the spectrum to allow the connexion between the metallicity and the substance-colour to be observed; whereas the substance-colour of metals is not known from direct observation, except, perhaps, in the case of gold, which in the state of gold-leaf lets through a greenish light.

I am now able to bring forward a striking confirmation of the relation which seems to exist between the light reflected as if from a metal, and that absorbed with great energy. On reading M. Haidinger's paper, of which the title has been already quoted, I was particularly interested by finding crystallized permanganate of potash mentioned among the substances which exhibit distinct surface- and substance-colours. I had previously noticed the very remarkable mode of absorption of light by red solution

as it is associated with only a colourless salt of potash, s green light with great energy, as is indicated by the even without the use of a prism. But when the light nitted by a pale solution is analysed by a prism, it is found here are five remarkable dark bands of absorption, or a of transparency, which are nearly equidistant, and are ed mainly in the green region. The first is situated on ositive or more refrangible side of the fixed line D, at ance, according to a measurement recently taken, of about evenths of the interval between consecutive bands; the last coincides with F. The first minimum is less conspicuous the second and third, which are the strongest of the set. t occurred to me, that as the solution is so opaque for rays the refrangibilities of these minima of transparency, cor- ding maxima might be expected in the light reflected from ystals. This expectation has since been realized by obser- s made on some small crystals. On analysing the reflected by a prism, I was readily able to observe four bright bands, xima, in the spectrum. These, as might have been expected, more easily seen when the light was incident nearly per- cularly than at a large angle of incidence. The first was yellow, the others green, passing on to bluish-green. ecomposing the light reflected at a considerable angle of nce, in a plane parallel to the axis, into two streams, zed respectively in, and perpendicularly to, the plane of nce, and analysing them by a prism, the bands were hardly t at all perceptible in the spectrum of the former pencil, that of the latter consisted of nothing but the bright bands. he tint alone of the first bright band already indicated that and was more refrangible than the light lying on the nega- ide of the first dark band seen in the spectrum of the light nitted by the solution, and less refrangible than the light between the first and second dark bands, so that its position correspond, or nearly so, to the first dark band. However, ye is greatly liable to be deceived, in experiments on absorp-

The sun's light was reflected horizontally into a darkened room, and allowed to fall on a crystal. The reflected light was limited by a slit, placed at the distance of two or three feet from the crystal. This precaution was taken to ensure making the observation on the regularly reflected light. Had no slit been used, or else a slit placed close to the crystal, it might have been supposed that the light observed was not regularly reflected, but merely scattered, as it would be by a coloured powder. The appearance of a spot of green light on a screen held at the place of the slit showed that the light was really regularly reflected. The slit was also traversed by the light scattered by the support of the crystal, etc. The slit was viewed through a prism and small telescope; and the position of the dark bands, or minima of brilliancy, in the reflected light could thus be compared with the fixed lines, which were seen by means of the scattered light in the uninterrupted spectrum corresponding to that portion of the slit through which the light reflected from the crystal did not pass. The minimum situated on the positive side of the first bright band lay at something more than a band-interval on the positive side of the fixed line D; the minimum beyond the fourth bright band lay at the distance of about half a band-interval on the negative side of F. It thus appears that the minima in the light reflected by the crystal were intermediate in position between the minima seen in the light transmitted through the solution, so that the maxima of the former corresponded to the minima of the latter.

It might have been considered satisfactory to compare the reflected light with the light transmitted, not by the solution, but by the crystals themselves. But the crystals absorb light with such energy as to be opaque; and even when they are spread out on glass, the film thus obtained is too deeply coloured for the purpose. For to show the bands well, the solution must be so dilute, or else seen through so small a thickness, as to be merely pink. As M. Haidinger states that the phenomena of the reflected light are the same for all the faces in all azimuths,

would be observed across a crystalline plate, were it possible to obtain one of sufficient thinness.

The first bright band in the reflected light does not usually appear to be very distinctly separated from the continuous light of lower refrangibility; but the latter may be got rid of by observing the light reflected about the polarizing angle, and analysing it so as to retain only the portion polarized perpendicularly to the plane of incidence. As the surface of the crystals is liable to become spoiled, it is safest in observations on the reflected light to make use of a crystal recently taken out of the mother-liquor. I have only observed four bright bands in the reflected light, whereas there are five distinct minima in the light transmitted by the solution. However, the extreme minima are less conspicuous than the intervening ones, besides which the fifth occurs in a comparatively feeble region of the spectrum. The fourth bright band in the reflected light was rather feeble, but with finer crystals perhaps even a fifth might have been visible. As the metallicity of the crystals is almost or perhaps quite insensible in the parts of the spectrum corresponding to the maxima of transparency, we may say, that, as regards the optical properties of the reflected light, the medium changes four or five times from a transparent substance to a metal and back again, as the refrangibility of the light changes from a little beyond the fixed line D to a little beyond F.

Extracts from Letter to Dr W. Haidinger: on the Direction of the Vibrations in Polarized Light: on Shadow Patterns and the Chromatic Aberration of the Eye: on Haidinger's Brushes. (*February* 9, 1854.)

Die Richtung der Schwingungen des Lichtäthers im polarisirten Lichte. Mittheilung aus einem Schreiben des Hrn. Prof. Stokes, nebst Bemerkungen von W. Haidinger.

[*Pogg. Ann.* XCVI, 1855, pp. 287—292. Mitgetheilt vom Hrn. Verf. aus d. *Sitzungsberichten d. Wiener Akademie* (*April* 1854).]

Ein Abschnitt des Schreibens vom Hrn. Prof. Stokes, den ich heute der hochverehrten mathematisch-naturwissenschaftlichen Classe vorzulegen die Ehre habe, bezieht sich auf die Richtung der Schwingungen des Lichtäthers in Bezug auf die Polarisations-Ebene, und zwar enthält er nicht nur eine Beurtheilung der Tragweite der Bemerkungen, welche ich als Beweis für die senkrechte Richtung dieser Schwingungen gegen diese Polarisations-Ebene aus den Erscheinungen an pleochromatischen Krystallen darstellen zu dürfen glaubte*, sondern auch seine Ansicht über den Gegenstand selbst, übereinstimmend mit seinen eigenen früheren Arbeiten....

"Die Thatsachen, deren ich in Bezug auf die Polarisation des Fluorescenz-Lichtes der Kalium-Platin-Cyanide (in einem andern Theile des Schreibens) gedachte, und die Art wie die Polarisirung der einfallenden Strahlen auf dieses Licht wirkt, stimmen, so viel ich glaube, viel besser mit der Annahme überein, dass die Schwingungen im polarisirten Lichte senk-

"Diess veranlasst mich, der Beweisgründe zu erwähnen, welche Sie anführten, um zu zeigen, dass im polarisirten Lichte die Schwingungen senkrecht auf der Polarisations-Ebene stehen. Da ich glaube, Sie würden gern meine Ansicht darüber kennen, so will ich sie ausführlich anführen. Zu allererst kann ich sagen, dass ich es nicht für *möglich* halte, durch *irgend* eine Combination von *anerkannten* Ergebnissen die Frage zu entscheiden. Unter den anerkannten Ergebnissen betrachte ich solche, wie diese — dass die Schwingungen transversal sind — dass im linear-polarisirten Lichte die Schwingungen geradlinig sind, und *symmetrisch* mit Beziehung der Polarisations-Ebene, und daher entweder parallel oder senkrecht auf diese Ebene — dass im elliptisch-polarisirten Lichte die Schwingungen elliptisch sind u. s. w. Die Entscheidung muss sich immer auf eine oder die andere Art auf dynamische oder physikalische Betrachtungen stützen, welche, mögen sie an sich noch so wahrscheinlich seyn, doch nicht zu den anerkannten Ergebnissen gezählt werden können. Es ist auch nicht schwierig zu sehen, welche die Betrachtungen dieser Art in dem Falle Ihrer Beweisführung sind. Nehmen wir den Fall eines doppelt absorbirenden einaxigen Krystalles, wie Turmalin. Es sey *oC* Fig. 6 parallel der Axe, *oA oB* zwei Richtungen senkrecht auf die Axe. Die eine Farbe (ich will sie *O* nennen) sieht man in der Richtung der Axe *Co*, und in allen Richtungen in der Ebene *BoA* (oder senkrecht auf die Axe) in dem *oC* parallel polarisirten Lichte. Die andere Farbe (*E*) sieht man in allen Richtungen in der Ebene *BoA*, wenn das Licht in dieser Ebene polarisirt ist, und man sieht sie gar nicht in der Richtung *Co*. 'Wenn diese Farbe nun von Transversal-Schwingungen abhängt, so sind alle solche Schwingungen, transversal oder senkrecht gegen die Axe, mit einem Male ausgeschlossen, und die einzigen Schwingungen, welche möglicherweise zu der Farbe des extraordinären Strahles, der in dem Krystall entsteht, gehören können, sind die parallel der Richtung der Axe.' Aber wenn von Schwingungen gesprochen wird, welche *zu* dieser oder jener Farbe *gehören*, so wird stillschweigend vorausgesetzt, dass in der That die Farbe abhängig ist von der Richtung der

dergestalt, dass sie als gegeben betrachtet werden kann, wenn die Richtung einer Linie gegeben ist, welche senkrecht auf den beiden vorerwähnten Richtungen steht. Nimmt man diesen Satz an in Bezug auf die Natur der Absorption, so ist vollkommen

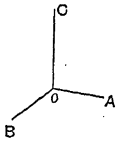


Fig. 6.

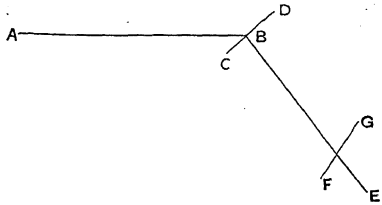


Fig. 7.

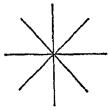


Fig. 8.

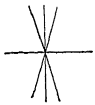


Fig. 9.

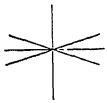


Fig. 10.

klar, dass die experimentalen Thatsachen, welche Sie in Bezug auf die doppelte Absorption anführen, zu dem Schlusse führen würde, die Schwingungen wären im polarisirten Lichte der Polarisations-Ebene parallel. Die Wahrscheinlichkeit, welche der Grund Ihrer Beweisführung der Wahrheit von Fresnel's Annahme giebt, reicht also nicht bis zur absoluten Gewissheit, sondern entspricht nur der Unwahrscheinlichkeit, dass Absorption gleichzeitig von der Richtung der Schwingungen und von der Richtung der Fortpflanzung abhängig seyn sollte, in der Art wie ich es oben erwähnte.

"Sie sagen: Wenn die Farbe *E*, welche man in der Richtu *Co* nicht sehen kann, irgendwie auf Transversal-Schwingun pn beruht, so sind alle solche Schwingungen transversal oder se m recht auf die Axe ausgeschlossen, und die einzigen Schwingun ie welche möglicherweise dem in dem Krystalle entstehenden ex kt, ordinären Strahl angehören können, sind dann der Richtung ne Axe parallel. Nun könnte aber eine besondere Farbe in ek-

auf das Licht, und das Licht besteht aus transversalen Schwingungen: sie könnte nicht von transversalen Schwingungen abhängen in der Richtung, wo sie nicht bloss von der Richtung der Schwingungen bestimmt ist, sondern auch von der Richtung der Fortpflanzung abhängt.

"Daher kann ich Ihre Folgerungen nicht als einen *Beweis* in dem strengen Sinne des Wortes betrachten. Ein solcher hängt am Ende von gewissen physikalischen Betrachtungen ab, welche sich auf die Absorption beziehen. Meine eigenen Ansichten in Bezug auf die Ursache der Absorption führen mich sehr stark zu der Meinung, dass sie bloss von der Richtung der Schwingungen und der Schwingungszeit (*periodic time*) und gar nicht von der Fortpflanzungsrichtung abhängt. In meinem Sinne haben daher Ihre Gründe sehr grosses Gewicht. Aber da diess von meinen eigenen individuellen Ansichten abhängt, so betrachte ich dieselben nicht als Etwas, was *nothwendiger* Weise zu allgemeiner Beistimmung zwingen muss.

"Da ich bei diesem Gegenstande bin, so erlauben Sie mir Ihre Aufmerksamkeit auf gewisse Untersuchungen zu lenken, welche mich in einer gänzlich verschiedenen Weise zu einer ähnlichen Schlussfassung führten. Sie sind in dem 9. Band der *Cambridge Philosophical Transactions, Part I*, veröffentlicht. Eine dynamische Untersuchung des Problems der Beugung, in anderen Worten eine mathematische Untersuchung der Beugung, behandelt wie ein dynamisches Problem, führte mich zu folgendem Gesetz: Wenn linear polarisirtes Licht der Beugung unterworfen wird, so ist jeder Strahl nach der Beugung linear polarisirt, und die Schwingungsebene des gebeugten Strahles ist parallel der Schwingungsrichtung des einfallenden Strahles. Unter Schwingungsebene ist die Ebene verstanden, welche durch den Strahl und durch die Schwingungsrichtung geht. Es sey *AB* Fig. 7 der Ebene des Papieres der einfallende Strahl, der bei gebeugt wird. *BE* auch in der Ebene des Papieres ein gebeugter Strahl, der in das Auge eintritt. *CD* in einer Ebene

die Schwingungsrichtung. Mit anderen Worten, die Schwingungsrichtung in dem gebeugten Strahle ist so nahe als möglich der Schwingungsrichtung in dem einfallenden Strahle parallel, als diess nur immer unter der Bedingung geschehen kann, dass sie senkrecht auf dem gebeugten Strahle steht. Dieses Gesetz erscheint sehr natürlich, selbst unabhängig von allem Calcül. Nun folgt aber aus demselben, dass wenn die Schwingungsebene zuerst mit der auf ABE senkrecht stehenden Ebene zusammenfällt, und dann allmählich durch gleiche Winkel herumgedreht wird, dass dann die Schwingungsebenen des gebeugten Strahles nicht gleichförmig ausgetheilt seyn werden, sondern sie werden mehr angehäuft gegen eine Ebene durch BE senkrecht auf die Ebene ABE erscheinen. Wenn α_i, α_d die Azimuthe der Schwingungsebene des einfallenden und des gebeugten Strahles sind, erhalten von Ebenen senkrecht auf ABE, und θ das Supplement des Winkels ABE, so haben wir $\tang \alpha_d = \cos\theta \tang \alpha_i$. Nun setzt uns aber der Versuch in den Stand die Richtung und das Maass jener Anhäufung der *Polarisationsebenen* zu bestimmen, und nach dem Ergebnisse werden wir uns geleitet finden, sie als parallel oder senkrecht auf die Schwingungsebenen zu betrachten. Wenn nun Fig. 8 die Projection der Polarisations-Ebenen des einfallenden Strahles auf einer senkrecht auf diesem Strahl stehenden Ebene in verschiedene Stellungen des Polarisirers (z. B. eines Nicol'schen Prismas in einer kreisförmig getheilten Fassung) darstellt, und Fig. 9 oder Fig. 10 dasselbe für den gebeugten Strahl vorstellt, so würden die Ebenen mehr gehäuft seyn wie in Fig. 9 und 10, je nachdem die Polarisations-Ebenen parallel oder senkrecht auf die Schwingungsebenen sind. Die Horizontallinien in Fig. 8, 9, 10 stellen die Projectionen auf der Ebene ABE dar. Bei einem Glasgitter geschieht die Beugung unter einem so bedeutenden Winkel, dass der theoretische Azimuth der Polarisations-Ebene des gebeugten Strahles in manchen Fällen bis zwanzig Grad variiren kann, je nachdem man voraussetzt, dass die Schwingungen des polarisirten Lichtes parallel oder senkrecht auf die Polarisations-Ebene stehen. Das Ergebniss der Versuche war vollständig

Mittheilung aus einem Schreiben des Hrn. Prof. Stokes, über das optische Schachbrettmuster; von W. Haidinger.

[*Pogg. Ann.* XCVI, 1855, pp. 305—312. Mitgetheilt vom Hrn. Verf. aus d. *Sitzungsberichten d. Wiener Akademie* (*April* 1854).]

....."Ich habe ähnliche Erscheinungen," wie das Interferenz-Schachbrettmuster, "auf einem Schirme dargestellt, indem ich das Sonnenlicht horizontal in ein finsteres Zimmer reflectirte, in dem Fenster, auf dem Wege des einfallenden Lichtes ein durchlöchertes Zinkblech, wie es für Fensterblenden dient, anbrachte, mit einer grossen Linse in einiger Entfernung von dem Bleche, und das Bild des Bleches nun auf einem Blatte Papier auffing, welches von dem Bilde nach beiden Seiten gegen die Linse zu und von derselben weg bewegt werden konnte. Ich überzeugte mich, dass die Erscheinung *nicht* auf Interferenz beruht, sondern einen viel einfacheren Charakter besitzt, und dass die Erklärung derselben aus der geometrischen Theorie der Schatten und Halbschatten folgt. In der That kenne ich kein Interferenz-Phänomen, das auf einer breiten Lichtfläche, wie die des Himmels ist, beruht; es ist immer erforderlich, die einfallenden Strahlen zu begränzen, indem sie etwa durch ein Loch oder einen Spalt gehen, oder indem man sich des Sonnenbildes einer Linse mit kurzer Brennweite bedient. In meinen Versuchen brachte ich nicht nur die Erscheinungen hervor, welche den von Ihnen beschriebenen ähnlich sind, indem ich den vollen Sonnenstrahl anwandte, sondern ich untersuchte auch die Interferenz-Wirkungen, indem ich das Bild der Sonne durch eine Linse von ziemlich kurzer Brennweite benützte und das Zinkblech nun in einige Entfernung vom Fenster rückte, und ich bin überzeugt, dass Interferenz-Wirkungen, selbst wenn sie nicht ganz unsichtbar seyn sollten, doch in Ihrem Phänomen so schwach sind, dass sie vernachlässigt werden können.

"Man betrachte zuerst Licht von einem einzigen Grade der Brechbarkeit (Fig. 11).

"Es seyen LL' die einfallenden Strahlen, SS' sey der durch-

als ob $\sigma\sigma'$ eine helle Scheibe wäre, von der die Strahlen ausgehen, von welchen uns aber nur diejenigen angehen, welche durch das Loch $\alpha\alpha'$ durchgehen ($\alpha\alpha'$ ist aber das Bild der Oeffnung), oder welche von $\sigma\sigma'$ in solchen Richtungen ausgehen, dass sie durch das Loch $\alpha\alpha'$ hindurchgehen würden, wenn man sie nicht durch den Schirm aufgefangen hätte. Es entsteht dadurch also, was man einen negativen Schatten $v\alpha\, v\alpha'$ und Halbschatten $A\alpha\sigma\, A'\alpha'\sigma'$ nennen könnte, das heisst Räume, welche für Beleuchtung eben dasjenige sind, was Schatten und Halbschatten für Finsterniss. Auf einem Schirme, mit welchem man die Strahlen auffängt, würde eine Kreisfläche beleuchtet seyn, am schmälsten bei $\alpha\alpha'$ (vorausgesetzt, dass $\sigma\sigma'$ grösser ist als $\alpha\alpha'$), und in dieser Entfernung auch gleichförmig hell, während bei anderen Entfernungen die Mitte heller seyn wird als der Rand.

"Man betrachte nun die Wirkung des Uebereinanderfallens der hellen Kreise, welche den benachbarten Oeffnungen entsprechen. Um die Frage auf das Aeusserste zu vereinfachen, nehme ich die Oeffnungen sehr klein an, so dass man $\alpha\alpha'$ als Punkt betrachten kann. Ich nehme dabei die Anordnung der Oeffnungen als die nämliche an, wie in der von Ihnen gegebenen Figur*. Stellt man den Schirm in den Focus, so erscheint eine Reihe heller Flecke (Fig. 13). Bewegt man den Schirm ein wenig in die Richtung gegen die Linse oder von derselben weg, so öffnen sich die lichten Flecke zu lichten Kreisflächen (Fig. 14). Es sey d die Entfernung zwischen den Mittelpunkten zweier benachbarter Kreise, in verticaler oder horizontaler Richtung. Bewegt man den Schirm so weit, bis die Radien der Kreise grösser sind als $\frac{1}{2}d$ aber kleiner als $d/\sqrt{2}$, so werden die Ränder der Scheiben über einander fallen, etwa so wie in Fig. 15. Der Schirm wird dadurch dem grössten Theile nach beleuchtet, mit Ausnahme von dunkeln quadratartigen, regelmässig geordneten Räumen (Fig. 16). Man bewege nun den Schirm so weit, bis die Radien der vergrösserten hellen Scheiben grösser sind als $d/\sqrt{2}$, aber noch immer kleiner als d, dann ist der Mittelpunkt der bisher dunklen Räume durch vier über einander fallende Kreisscheiben beleuchtet (Fig. 17), während der Mittelpunkt

mes sind also nun in regelmässiger Anordnung die Punkte wie a, welche *in ihrer Lage denjenigen Gegenden des Schirmes entsprechen, welche, wenn dieser im Focus stand, die Mittelpunkte*

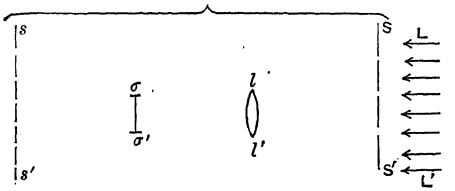


FIG. 11.

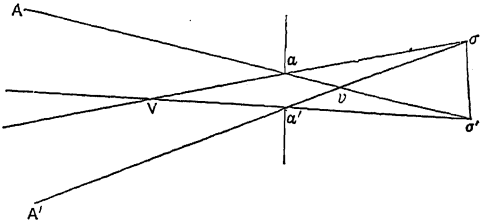


FIG. 12.

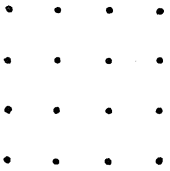


FIG. 13.

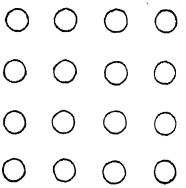


FIG. 14.

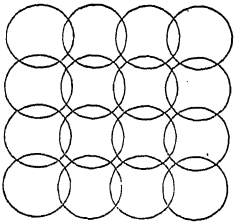
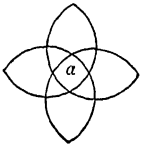
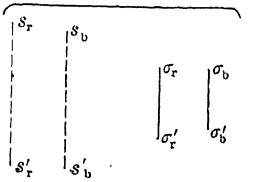


FIG. 15.





man dann nicht helle Punkte auf dunklem Felde, sondern dunkle Punkte auf hellem Felde hat, wobei die dunkeln Punkte in ihrer Lage den Mittelpunkten der Scheiben entsprechen, oder jenen Punkten, welche hell waren, als sich der Schirm im Focus befand.

"Im Allgemeinen wird das gleiche Ergebniss folgen, auch wenn die Oeffnungen nicht ganz klein sind aber doch noch in einigem Verhältnisse zu den Zwischenräumen stehen. Doch dürften die verschiedenen Phasen der Erscheinung in diesem Falle sich weniger auffallend darstellen, als in jenem.

"Betrachten wir nun die Wirkung der Ueberlagerung der verschiedenen Bestandtheile des weissen Lichtes. Anstatt des einfachen Bildes der Sonne $\sigma\sigma'$ und des durchlöcherten Schirmes ss' haben wir eine unendliche Menge von Bildern (Fig. 18), von welchen die stärker gebrochenen, wie $\sigma_b\sigma'_b$, $s_b s'_b$, näher an der Linse liegen als die weniger gebrochenen. Da der Schirm, auf welchem das Bild aufgefangen wird, ziemlich nahe an dem Orte der deutlichsten Erscheinung des durchlöcherten Schirmes aufgestellt ist, so werden die chromatischen Abweichungen des Bildes ss' viel weniger wichtig seyn, als die von $\sigma\sigma'$, und sie mögen daher hier der Einfachheit wegen gänzlich übergangen werden. Wird nun also der Papierschirm aus seiner früheren Stellung in Brennpunkte von der Linse hinweggerückt, so folgen sich die verschiedenen Phasen der Erscheinung schneller für die mehr als für die weniger brechbaren Farben, und diess aus dem Grunde, weil der Schirm von $s_b s'_b$ weiter absteht, als von $s_r s'_r$. Wenn dagegen der Schirm gegen die Linse zu bewegt wird, so geschehen die Aenderungen früher für die weniger als für die mehr brechbaren Farben.

"Das gleiche Princip erklärt auch die Erscheinung im Auge. Die Hornhaut, Krystall-Linse u. s. w. nehmen den Platz der Linse ein, die Netzhaut ersetzt den Papierschirm. Die Hauptverschiedenheit liegt in der Art, wie der durch jede einzelne Oeffnung kommende Strahlenbündel begränzt ist. In dem oben betrachteten Falle war er begränzt durch oder in Folge der

Schirm (die Netzhaut), welcher fest steht, während die Bilder *ss'* bewegt werden, in Folge der Bewegung des Gegenstandes, der diese Bilder hervorbringt, eine Bewegung, welche in allen Fällen von Brechung eine Bewegung des Bildes in derselben Richtung zur Folge hat. Aber keiner dieser beiden Umstände hat einen Einfluss auf einen Erklärungsgrund.

"Man halte nun ein Stück durchlöcherte Karte oder durchlöchertes Papier gegen den Himmel, in der Entfernung der deutlichsten Sehweite, und nähere es dann allmählich dem Auge. Die wahren Bilder der Karte, welche den verschiedenen Farben entsprechen, fallen nun hinter die Netzhaut; da aber die mehr gebrochenen Bilder vor den weniger gebrochenen liegen, so sind sie weniger ausserhalb des Brennpunktes. Daher finden die Veränderungen der Erscheinung schneller statt für die weniger als für die mehr brechbaren Farben. Wenn daher die dunkeln Zwischenräume in die dunkeln Flecken überzugehen beginnen, so sind sie roth umsäumt, weil die rothen Kreisscheiben auf der Netzhaut grösser sind als die blauen. Diese Umsäumung durch Roth, oder vielmehr durch die mehr brechbaren Farben in ihrer Folge, könnte vielleicht zu wenig lebhaft seyn, um einen Eindruck hervorzubringen. Wenn durch das Uebereinanderfallen der Kreise die dunkeln Flecke in helle Flecke verwandelt wurden, so sind die letzteren gelblich von dem Vorwalten der weniger brechbaren Farben, während das allgemeine Feld blaulich ist, von dem Vorwalten der mehr brechbaren. Wird die durchlöcherte Karte, aus der früheren Stellung in der Entfernung des deutlichsten Sehens in eine grössere Entfernung vom Auge gerückt, so liegen die von den verschiedenen im weissen Licht enthaltenen Farben herrührenden Bilder der Karte vor der Netzhaut, zu äusserst die mehr brechbaren, und sie sind daher entfernter vom Brennpunkte als die weniger brechbaren. Daher sind die Farben der Flecken und Zwischenräume die entgegengesetzten von denen in der früheren Lage.

"Ich bemerke hier, dass das Auge nicht achromatisch ist. Schon Fraunhofer hat diess in seinen Bemerkungen über das

wie ein Licht oder die Sonnenscheibe, durch ein tiefblaues Glas oder durch eine Verbindung mehrerer solcher Gläser betrachtet, welche keinen anderen sichtbaren Strahlen den Durchgang gestatten ausser den äussersten rothen und violetten, so sieht man die rothen und die violetten Bilder der Gegenstände nicht gleich deutlich zusammen. Wenn ich die Sonnenscheibe durch eine Combination dieser Art betrachte, was ohne die geringste Unbequemlichkeit ausführbar ist, wenn man nur ein hinlänglich dunkles Glas oder eine hinlängliche Anzahl von Gläsern anwendet, so sehe ich eine wohl begränzte rothe Scheibe und eine undeutliche violette Scheibe von etwa dem doppelten Durchmesser der ersteren. Die letztere kann durch die Anwendung einer convexen Linse deutlich gemacht werden, aber dann wird jene andere undeutliche. In der That kann ich entfernte Gegenstände deutlich vermittelst der äussersten rothen Strahlen sehen, bin aber entschieden kurzsichtig in Bezug auf die violetten Strahlen. Für mittlere Strahlen, und übereinstimmend für gewöhnliches Licht, sollte ich daher etwas weniger kurzsichtig seyn, welches auch der Fall ist."

Einige neuere Ansichten über die Natur der Polarisationsbüschel; von W. Haidinger.

[*Pogg. Ann.* XCVI, 1855, p. 314. Mitgetheilt vom Hrn. Verf. aus d. *Sitzungsberichten d. Wiener Akademie* (*Mai* 1854).]

"Ich bin keinesweges durch irgend welche der Erklärungsarten befriedigt, welche ich bisher über die Ursache Ihrer Büschel gesehen habe. Man kann allen, vorzüglich aber der des Hrn. Jamin, einen Einwurf machen, der unwiderlegbar scheint. Ich will diesen Gegenstand aber hier nicht weiter verfolgen, weil ich daran bin demnächst einen Aufsatz darüber an das *Philosophical Magazine* zu schicken. Ich bin überzeugt, dass die Erscheinung entweder in oder knapp an der Netzhaut ihren Sitz hat. Ich werde eine Muthmassung in Bezug auf die Ursache derselben aufstellen, nach welcher sie von der Art abhängen, wie die letzten Nervenfasern die Empfindung des Lichtes aufnehmen.

On the Theory of the Electric Telegraph.

By Prof. W. Thomson.* (Extract.)

[From the *Proceedings of the Royal Society*, May 1855: also Lord Kelvin's *Mathematical and Physical Papers*, II, pp. 61–76.]

Extract of Letter from Prof. Stokes to Prof. W. Thomson (dated Nov. 1854).

"In working out for myself various forms of the solution of the equation $\frac{dv}{dt}=\frac{d^2v}{dx^2}$ under the conditions $v=0$ when $t=0$ from $x=0$ to $x=\infty$; $v=f(t)$, when $x=0$ from $t=0$ to $t=\infty$, I found that the solution with a single integral only (and there must necessarily be this one) was got out most easily thus:—

"Let v be expanded in a definite integral of the form

$$v=\int_0^\infty \varpi(t, \alpha)\sin \alpha x\, dx,$$

which we know is possible.

"Since v does not vanish when $x=0$, $\frac{d^2v}{dx^2}$ is not obtained by differentiating under the integral sign, but the term $\frac{2}{\pi}\alpha v_{x=0}$ must be supplied†, so that (observing that $v_{x=0}=f(t)$ by one of the equations of condition) we have

$$\frac{d^2v}{dx^2}=\int_0^\infty \left\{\frac{2}{\pi}\alpha f(t)-\alpha^2\varpi\right\}\sin \alpha x\, dx.$$

Hence

$$\frac{dv}{dt}-\frac{d^2v}{dx^2}=\int_0^\infty \left\{\frac{d\varpi}{dt}+\alpha^2\varpi-\frac{2}{\pi}\alpha f(t)\right\}\sin \alpha x\, dx,$$

[* This investigation was commenced in consequence of a letter received by the author from Prof. Stokes dated Oct. 16, 1854, and consists mainly of two letters

and the second member of the equation being the direct development of the first, which is equal to zero, we must have

$$\frac{d\varpi}{dt} + \alpha^2\varpi - \frac{2}{\pi}\alpha f(t) = 0,$$

whence

$$\varpi = \epsilon^{-\alpha^2 t}\int_0^t \frac{2}{\pi}\alpha f(t)\,\epsilon^{\alpha^2 t}\,dt,$$

the inferior limit being an arbitrary function of α. But the other equation of condition gives

$$\varpi = \epsilon^{-\alpha^2 t}\int_0^t \frac{2}{\pi}\alpha f(t)\,\epsilon^{\alpha^2 t}\,dt = \left(\frac{\pi}{2}\right)^{-1}\alpha\int_0^t \epsilon^{-\alpha^2\overline{t-t'}} f(t')\,dt',$$

therefore

$$v = \left(\frac{\pi}{2}\right)^{-1}\int_0^\infty\int_0^t f(t')\,\alpha\epsilon^{-\alpha^2\overline{t-t'}}\sin\alpha x\,d\alpha\,dt'.$$

But

$$\int_0^\infty \epsilon^{-a\alpha^2}\cos b\alpha\,d\alpha = \frac{1}{2}\left(\frac{\pi}{a}\right)^{\frac{1}{2}}\epsilon^{-\frac{b^2}{4a}},$$

therefore

$$\int_0^\infty \epsilon^{-a\alpha^2}\sin b\alpha\,.\,\alpha\,d\alpha = -\frac{d}{db}\left\{\frac{1}{2}\left(\frac{\pi}{a}\right)^{\frac{1}{2}}\epsilon^{-\frac{b^2}{4a}}\right\}$$

$$= \frac{\pi^{\frac{1}{2}}b}{4a^{\frac{3}{2}}}\,\epsilon^{-\frac{b^2}{4a}},$$

whence writing $t-t'$, x, for a, b, and substituting, we have

$$v = \frac{x}{2\pi^{\frac{1}{2}}}\int_0^t (t-t')^{-\frac{3}{2}}\,\epsilon^{\frac{x^2}{4(t-t')}} f(t')\,dt'.$$

"Your conclusion as to the American wire follows from the differential equation itself which you have obtained. For the equation $kc\dfrac{dv}{dt} = \dfrac{d^2v}{dx^2}$ shows that two submarine wires will be similar, provided the squares of the lengths x, measured to similarly situated points, and therefore of course those of the whole lengths l, vary as the times divided by ck; or the time of any electrical operation is proportional to kcl^2.

On the Achromatism of a Double Object-glass.

[From the *Report of the British Association*, Glasgow, 1855, pp. 14–15.]

The general theory of the mode of rendering an object-glass achromatic by combining a flint-glass with a crown-glass lens, is well known. The achromatism is never perfect, on account of the irrationality of dispersion. The defect thence arising cannot possibly be obviated, except by altering the composition of the glass. It seemed worthy of consideration whether much improvement might not be effected in this direction; but the problem which the author proposed for consideration was only the following:—Given the kinds of glass to be employed, to find what ought to be done so as to produce the best effect; in other words, to determine the ratio of the focal lengths which gives the nearest approach to perfect achromatism. Two classes of methods may be employed for this purpose. In the one, compensations are effected by trial on a small scale; in the other, the refractive indices of each kind of glass are determined for certain well-defined objects in the spectrum, such for example as the principal fixed lines. The former has this disadvantage, that compensations on a small scale do not furnish so delicate a test as the performance of a large object-glass. The observation of refractive indices, on the other hand, admits of great precision; but it does not immediately appear what ought to be done with the refractive indices when they are obtained. After alluding to the method proposed by Fraunhofer for combining the refractive indices, which, however, as he himself remarked, did not lead to results in exact accordance with observation, the author proposed the following as the con-

The refractive index of the flint-glass may be regarded as a function of the refractive index of the crown-glass, and may be expressed with sufficient accuracy by a series with three terms only. The three arbitrary constants may be determined by the values of three refractive indices determined for each kind of glass. The result is as follows:—Let μ_1, μ_2, μ_3 be the refractive indices for the crown-glass; μ_1', μ_2', μ_3' the same for the flint-glass; μ, μ' the refractive indices of the two glasses for any arbitrary ray; m the value of μ for the point at which the focal length is to be made a minimum; r the ratio of $\Delta\mu'$ to $\Delta\mu$ to be employed in the ordinary formula for achromatism. Then having calculated numerically

$$r_{1,2} = \frac{\mu_2' - \mu_1'}{\mu_2 - \mu_1}, \qquad r_{2,3} = \frac{\mu_3' - \mu_2'}{\mu_3 - \mu_2},$$

we shall have*

$$r = r_{1,2} + \frac{2m - \mu_1 - \mu_2}{\mu_3 - \mu_1}(r_{2,3} - r_{1,2}).$$

For the value of m it will be sufficient to take

$$\mu_D + \tfrac{1}{3}(\mu_E - \mu_D).$$

On applying this formula to calculate r for the object-glass for which Fraunhofer has given both the refractive indices of the component glasses and the value of r, which, as observation showed, gave the best results, and taking in succession various combinations of three lines each out of the seven used by Fraunhofer, the author found that whenever the combination was judiciously chosen, the resulting value of r was the same, whatever might have been the combination, and equal to 1·980, which is precisely the value determined by Fraunhofer from observation, as giving the best effect†.

[* In fact the value of the relative dispersion r for the ray of index μ may be obtained from $r = d\mu'/d\mu$, where $\mu' = A + Bx + Cx^2$, in which x represents $\mu - \mu_2$.]

[† The subject of this note is resumed and amplified in a lecture "On the Principles of the Chemical Correction of Object-glasses" delivered to the Photographic Society, and published in the *Photographic Journal*, Feb. 15, 1873;

Remarks on Professor Challis's paper, entitled "A Theory of the Composition of Colours etc."

[From the *Philosophical Magazine*, XII, 1856, pp. 421–5.]

My object in the present communication is not to discuss Professor Challis's theory, but to rectify some statements as to the experimental facts of the case, as well as one relating to the extent of some researches of my own. I have, however, on some points expressed opinions, respecting the justice of which it is only one who is familiar with certain classes of optical experiments who can feel the confidence that I entertain.

From the paragraph commencing at the foot of page 330, it is plain that Professor Challis has made some confusion between three perfectly distinct things: Sir David Brewster's controverted analysis of the solar spectrum by means of absorbing media*; his discovery of the phenomenon of internal dispersion†; and my own discovery, that a beam of rays of prismatic purity (whether belonging to the visible or invisible portion of the spectrum is indifferent) may, by their action on certain media, produce light which may be decomposed by the prism into portions extending over a wide range of refrangibility, and having colours answering to their refrangibilities‡.

As to the first, it was asserted by Sir David Brewster that light of prismatic purity may have its colour changed by passing through absorbing media. This has nothing to do with "internal" or "epipolic" dispersion, or "fluorescence." Glass coloured blue by cobalt, for instance, has none of these properties, although it

made out, it would be a point of the utmost importance to consider in reference to any physical theory of light. But while none deny that the appearances are as stated by Sir David Brewster, the inference to be drawn from those appearances remains open to discussion. Airy*, Helmholtz†, and Bernard‡, by operating in a different manner, have come to the conclusion that the colour is not changed; and Helmholtz has attributed the apparent change partly to the mixture of a very small quantity of stray light, partly to the effects of contrast. Having been much in the habit of analysing the light transmitted by coloured solutions, and having repeatedly seen the phenomena on which Sir David Brewster relies, I may be permitted to express my belief that the change of colour is only apparent, being an illusion depending upon contrast, and that this is one of the cases in which the direct evidence of the senses must be controlled. Were the change of colour real, Prof. Challis's statement (p. 330), that "experiment has proved that both the colour and the angle of refraction for a given angle of incidence depend, the substance being given, only on the value of λ," would cease to be true.

As to the second, the principal phenomenon consists in this: that when a beam of sunlight, condensed by a lens, is admitted into certain perfectly clear (*i.e.* not muddy) media, the path of the rays is marked by light, of different colours in different cases, which emanates in all directions. As the real nature of this remarkable phenomenon was not at the time understood, and the phenomenon itself was confounded with the effects of mere suspended particles, it is needless to discuss its possible bearing on any theory of the sensation of colour under this head.

As to the third, the new light emanating from the media which possess the property in question is just like any other light of the same prismatic composition. In its physical properties it retains no traces of its parentage, and its colour depends simply upon its new refrangibility, having nothing to do with that of

to the visible or the invisible part of the spectrum. Hence, in speculating on the sensation of colour, this phenomenon may be set aside as not bearing upon the question. I may remark, however, that with regard to the sensation of colour, an analogy has often struck me between the retina and a fluorescent substance, or rather a mixture of three or more fluorescent substances: but this is only an analogy.

It is not true, as Professor Challis seems to suppose (p. 332), that absorption is always, or even generally, accompanied by epipolic dispersion. Among the great variety of coloured metallic solutions, I have hitherto found that property only in solutions of salts of sesquioxide of uranium. I make this remark merely by the way, to prevent misconception: I perfectly agree with Professor Challis in believing that a ray of definite refrangibility is uncompounded; in fact, it was my firm belief in that doctrine which led me to make out the phenomenon of the change of refrangibility of light.

The superposition of two coloured glasses or ribbons by no means gives the effect of the mixture of the two colours. Various methods of mixing colours are enumerated by Mr Maxwell at the end of his paper, entitled "Experiments on Colour, etc.," in the twenty-first volume of the *Edinburgh Transactions*, p. 275. The production of white by a mixture of blue and yellow is by no means confined to prismatic blue and yellow, but takes place just as well with the colours of coloured bodies. In making experiments with the spectrum, in order to neutralize, when possible, a prismatic colour of given intensity by another prismatic colour, so as to produce white, *two* points must be attended to: the place of the second colour in the spectrum must be properly chosen, and the intensity of the light properly regulated. Hence any speculations as to the cause of the variations of intensity in the solar spectrum can have no bearing on the subject before us, seeing that the relation between the intensities of the mixed colours necessary for the production of whiteness is a matter of experimental adjustment.

manner of Sir John Herschel, in which the abscissa x denotes refrangibility, measured, suppose, by the distance from the extreme red in some standard spectrum, and the ordinate y denotes the intensity; so that ydx is the quantity of light between the refrangibilities x and $x+dx$, the intensity in the incident light being taken equal to unity, for simplicity's sake, whatever be the value of x, as we only care to compare intensities for the same value of x. Let y, y' be the ordinates in the curves belonging to two glasses, y_s the ordinate belonging to the tint obtained by superposing the glasses, y_m the ordinate belonging to the mixed tint, as procured, for instance, by a double-image prism, in which case each of the superposed differently coloured images has half the brightness of the original. Then $y_s = yy'$, but $y_m = \frac{1}{2}(y+y')$; and it is easy to see how different may be the curves whose ordinates are y_s, y_m respectively. Thus, let the scale of abscissæ be such that the spectrum extends from $x=0$ to $x=\pi$, and let $y=\frac{1}{4}(1-\cos x)^2$, $y'=\frac{1}{4}(1+\cos x)^2$. In this case $y_s=\frac{1}{16}\sin^4 x$, which vanishes at the extremities, and is a maximum in the middle; whereas $y_m=\frac{1}{4}(1+\cos^2 x)$, which is a maximum at the two extremities, and a minimum in the middle. In the former case, the tint would be a sort of green, a pretty full colour; in the latter, a sort of dilute purple. The colours of two ribbons may very conveniently be mixed in equal proportion by placing them side by side, and viewing them through a double-image achromatic prism; and it will be seen how different the mixed colour is from that seen on superposing the ribbons and holding them up to the light.

I cannot agree with Professor Challis, that "the coloured light of substances, though derived from sunlight, is in fact new light," except so far as relates to that portion which arises from fluorescence. But fluorescence is often absent altogether; and even when it exists, the colour thence arising must in most cases be but a small fraction of the whole colour observed when the substance is freely exposed to white light, not viewed under absorbing media. I think that any one who has been in the

small number of cases in which the colour observed is referable to other causes, the colours of natural bodies are due to absorption. The exceptions are colours due to fluorescence, as in the case of solutions of quinine, or to regular chromatic reflexion, as in the case of gold, copper, platino-cyanide of magnesium, murexide, etc., not to mention such colours as those of the rainbow, etc., which result from the general properties of bodies with regard to their action on light, not from any speciality of the substance by which the colours happen to be produced. The mode in which I conceive absorption to operate in occasioning the colours observed in dyed ribbons, flowers, coloured powders, etc., I have more fully explained elsewhere*. Now absorption is best studied in clear solids or solutions, where it is not complicated by irregular reflexions or refractions. But when such media are studied by the aid of a pure spectrum, there cannot be a moment's hesitation that the colour of the transmitted light is due to the abstraction from the incident white light of some of the component rays, as explained by Newton. The colour results, not from the light acted on by the medium, but precisely from the portion left unaffected. Hence its origin is celestial (supposing the sun to be the source of the light employed), not terrestrial. But if the colours of natural bodies arise from absorption, the origin of those colours must be deemed celestial too. To make the origin of the green colour of a leaf terrestrial, but that of the green colour of the light transmitted through an alcoholic solution of the colouring matter celestial, notwithstanding that the two greens agree in their very remarkable prismatic composition, would be needlessly and most capriciously to multiply the causes of natural phenomena. The light which gives us the sensation of greenness when we look at a leaf is, I conceive, no more terrestrial in its origin than the sun's light reflected from a mirror is terrestrial, as not retaining the direction which it had in travelling to us from the sun. It is only in the phenomenon of fluorescence, and the closely allied phenomenon of phosphorescence, that the light emitted can be considered as new light having a terrestrial origin.

EMENT TO THE "ACCOUNT OF PENDULUM EXPERIMENTS NDERTAKEN IN THE HARTON COLLIERY...". BY G. B. IRY, Esq., Astronomer Royal.

om the *Philosophical Transactions* for 1856. Received *Feb.* 13, read *Mar.* 6, 1856.]

ADDENDUM (pp. 353–355).

communicating with Professor Stokes, in reference to the of the Earth's rotation and ellipticity in modifying the ical results of the Harton Experiment, I was favoured by entleman with an investigation, which, with his permission, oin as a valuable addition to my own paper.

shall suppose the surface of the Earth to be an ellipsoid olution, and will employ the notation made use of in my on Clairaut's Theorem, published in the fourth volume of *mbridge and Dublin Mathematical Journal* *. In this,

is the potential of the Earth's mass.

θ are the polar coordinates of any point in or exterior to rth's surface; r being measured from the centre, and θ from is of rotation.

is the equatorial radius.

he ellipticity.

the angular velocity.

the ratio of the centrifugal force to gravity at the equator.

the mass of the Earth.

he angle between the normal and radius vector at any point surface.

If
$$U = V + \frac{\omega^2}{2} r^2 \sin^2 \theta,$$
the differential coefficients
$$\frac{dU}{dr}, \quad \frac{1}{r}\frac{dU}{d\theta}$$
will give the components of the force along and perpendicular to the radius vector; and, g being the force of gravity,
$$g = -\cos\nu \frac{dU}{dr} + \sin\nu \frac{1}{r}\frac{dU}{d\theta};$$
which becomes, since ν and $\frac{dU}{d\theta}$ are small quantities of the first order,
$$g = -\frac{dU}{dr}.$$
Let v be measured along the vertical; then
$$\frac{dg}{dv} = \cos\nu \frac{dg}{dr} - \sin\nu \frac{1}{r}\frac{dg}{d\theta},$$
or, to the first order,
$$\frac{dg}{dv} = \frac{dg}{dr} = -\frac{d^2U}{dr^2}.$$

Let c be the depth of the mine; then if $(c/a)^2$ be neglected, we shall have for the value of the fraction $\frac{\text{gravity below}}{\text{gravity above}}$ (which I will call F), calculated on the supposition that all the attracting mass is internal to both stations,
$$F = 1 - \frac{c}{g}\frac{dg}{dv},$$
where, after differentiation, r is to be put equal to the radius vector of the surface, namely $a(1 - \epsilon\cos^2\theta)$. Now the value of V (Article 5 of the paper referred to) is
$$V = \frac{E}{r} - \left(\frac{E\epsilon}{a} - \frac{1}{2}\omega^2 a^2\right)\frac{a^3}{r^3}\left(\cos^2\theta - \frac{1}{3}\right),$$
which is true independently of any particular hypothesis re-

and $$g = -\frac{dU}{dr}$$

$$= \frac{E}{r^2} - 3\left(\frac{E\epsilon}{a} - \frac{1}{2}\omega^2 a^2\right)\frac{a^3}{r^4}\left(\cos^2\theta - \frac{1}{3}\right) - \omega^2 r \sin^2\theta,$$

whence $$-\frac{dg}{dv} = -\frac{dg}{dr}$$

$$= \frac{2E}{r^3} - 12\left(\frac{E\epsilon}{a} - \frac{1}{2}\omega^2 a^2\right)\frac{a^3}{r^5}\left(\cos^2\theta - \frac{1}{3}\right) + \omega^2 \sin^2\theta.$$

Putting now $r = a(1 - \epsilon\cos^2\theta)$, $\omega^2 = m\dfrac{E}{a^3}$, we find

$$g = \frac{E}{a^2}(1 + 2\epsilon\cos^2\theta) - \frac{3E}{a^2}\left(\epsilon - \frac{m}{2}\right)\left(\cos^2\theta - \frac{1}{3}\right) - m\frac{E}{a^2}(1 - \cos^2\theta)$$

$$= \frac{E}{a^2}\left\{1 + \left(\frac{5m}{2} - \epsilon\right)\cos^2\theta + \epsilon - \frac{3m}{2}\right\},$$

$$-\frac{dg}{dv} = \frac{2E}{a^3}(1 + 3\epsilon\cos^2\theta) - \frac{12E}{a^3}\left(\epsilon - \frac{m}{2}\right)\left(\cos^2\theta - \frac{1}{3}\right) + \frac{mE}{a^3}(1 - \cos^2\theta)$$

$$= \frac{2E}{a^2}\left\{1 + \left(\frac{5m}{2} - 3\epsilon\right)\cos^2\theta + 2\epsilon - \frac{m}{2}\right\}.$$

Whence

$$-\frac{1}{g}\frac{dg}{dv} = \frac{2}{a}\left\{\begin{array}{l} 1 + \left(\dfrac{5m}{2} - 3\epsilon\right)\cos^2\theta + 2\epsilon - \dfrac{m}{2} \\ \quad - \left(\dfrac{5m}{2} - \epsilon\right)\cos^2\theta - \epsilon + \dfrac{3m}{2} \end{array}\right\}$$

$$= \frac{2}{a}\{1 - 2\epsilon\cos^2\theta + \epsilon + m\};$$

and therefore

$$F = 1 + \frac{2c}{a}\{1 - 2\epsilon\cos^2\theta + \epsilon + m\}.$$

Now the method adopted in the 'Account of Experiments,' etc., Article 57, gives

Therefore, if R be the ratio of the value of $F-1$ given above, to $F-1$ as calculated by the method of the 'Account of Experiments,'

$$R = \frac{1 - 2\epsilon \cos^2 \theta + \epsilon + m}{1 + \epsilon \cos^2 \theta} = 1 - 3\epsilon \cos^2 \theta + \epsilon + m.$$

If l be the geocentric latitude of the place, we may in the small term replace θ by $90° - l$; and since

$$\cos^2 \theta = \sin^2 l = \tfrac{1}{2}\,(1 - \cos 2l),$$

we find

$$R = 1 + m - \frac{\epsilon}{2} + \frac{3\epsilon}{2} \cos 2l.$$

Now

$$m = \tfrac{1}{289} = 0{\cdot}00346$$

$$\epsilon = \frac{1}{300{\cdot}8} = 0{\cdot}00333$$

$$l, \text{ for Harton}, = 54°\ 48';$$

$$R = 1 + 0{\cdot}00346 - 0{\cdot}00334 = 1{\cdot}00012.$$

That R should have been so very nearly equal to unity, depends upon an accidental numerical relation between the values of m, ϵ, and l. At the equator, $R-1$ would have been as great as 0·00679.

In Article 60 of the 'Account,' $F-1$ was found $= {\cdot}00012032$; whence $R(F-1) = {\cdot}00012033$; which only alters the final value of the mean density in the ratio of 6836 to 6835, giving for result

6·565.

At the equator, the correction to the deduced value 6·566 would have been $-{\cdot}077$."

On the Polarization of Diffracted Light.

[From the *Philosophical Magazine*, XIII, 1857, pp. 159–61: also *Pogg. Ann.* CI, 1857, pp. 154–7.]

On considering the recent interesting experimental researches of M. Holtzmann on this subject*, I am induced to make the following remarks†.

In the more common phenomena of diffraction, in which the angle of diffraction is but small, we know that the character of the diffracting edge, and the nature of the body by which the light is obstructed, are matters of indifference. This was made the object of special experimental investigation by Fresnel; and its truth is further confirmed by the wonderful accordance which he found between the results of the most careful measurements and the predictions of a theory in which it is assumed that the office of the opaque body is merely to stop a portion of the incident light. But when diffraction is produced by a fine grating, the angle of diffraction is no longer restricted to be small; and it becomes an open question whether the precise circumstances of the diffraction may not have to be taken into account, and not merely the form and dimensions of the apertures through which the light passes. If so, the problem becomes one of extreme complexity. In my memoir on the Dynamical Theory of Diffraction, published in the ninth volume of the *Cambridge Philosophical Transactions*, I investigated the problem on the hypothesis that in diffraction at a large angle, as we know to be the case in diffraction at a small one, the office of the opaque body is *merely to stop* a portion of the incident light. I distinctly stated this as a hypothesis, and I always regarded it as rather precarious. I was guided by the following consideration. Let AB

be the section of a transparent interval by the plane of diffraction, supposing for simplicity the diffraction to take place in air or in a homogeneous medium, and not at the confines of two different media; let $AB = b$; let β be the angle of diffraction, and λ the wave-length in the medium. Supposing the light to be incident perpendicularly on the grating, the difference of phase of the secondary waves which started from A, B, respectively, will be determined by the length of path $b \sin \beta$ within the medium. In experiment this will usually be a considerable multiple of λ. In the line AB take two points, A', B', equidistant from A, B, respectively, and comprising between them as large a multiple as possible of $\lambda \operatorname{cosec} \beta$. If we suppose the influence of the opaque body insensible at the distance AA' or BB' from A or B, the secondary waves which start from all points in the interval $A'B'$ will neutralize each other by interference, so that the whole effect will be due to the secondary waves which start from AA' and BB'. Suppose the angle β to belong to the brightest part of a "spectrum of the first class" (Fraunhofer); then $AA' + BB' = \frac{1}{2}\lambda \operatorname{cosec} \beta$, λ referring to mean rays, so that AA' or BB' is only equal to $\frac{1}{4}\lambda \operatorname{cosec} \beta$. If, for example, $\beta = 30°$, AA' is only equal to $\frac{1}{2}\lambda$. At such very small distances it may well be doubted whether the influence of the opaque body may not have to be taken into account.

When diffraction takes place at the confines of two different media, suppose air and glass, the problem is still further complicated. We may, however, apply the theory to which reference has been made on the two extreme suppositions, first, that the diffraction takes place wholly in the first, secondly, that it takes place wholly in the second medium. The results of my own experiments were very fairly represented by theory, the vibrations being supposed perpendicular to the plane of polarization, provided the diffraction be conceived to take place in the *first* medium, or in other words, just *before* the light reaches the grating; but they would not at all fit the hypothesis of vibrations parallel to the plane of polarization. I put forth some

argument in favour of the hypothesis that the vibrations are perpendicular to the plane of polarization, though I still felt the necessity of repeating the experiments under varied circumstances.

But since the appearance of M. Holtzmann's researches the state of the question is changed. I have no reason to doubt the correctness of his results, while on the other hand the result I myself obtained was far too decided to be passed by. The conclusion which, in the present state of the question, seems to me most probable is, that the polarization of light diffracted at a large angle is, in fact, influenced by the nature of the diffracting body. The subject demands a much more extensive experimental investigation, in which the circumstances of diffraction shall be varied as much as possible. I hope to have leisure to undertake such an investigation: meanwhile it would be premature to offer any decided opinion. It seems to me, however, worthy of attentive consideration, whether a glass grating may not offer a fairer experiment for the decision of the question as to the direction of vibration in polarized light than a smoke grating, inasmuch as in the former we have to do with an uninterrupted medium, glass, the surface of which is merely rendered irregular, whereas in the latter the problem is complicated by the existence of two distinct media, glass and soot, placed alternately. I call the layer of soot a medium, for though no light can pass through any sensible thickness of it, we must not conclude from that that it is without influence on the light which passes excessively close to it.

I have not mentioned the effect of oblique refraction in the experiments of M. Holtzmann, because if it were allowed for, the character of the results obtained would remain unchanged, the magnitude of the observed effect would only be somewhat diminished.

On the Discontinuity of Arbitrary Constants which appear in Divergent Developments.

[From the *Transactions of the Cambridge Philosophical Society*, Vol. x, pp. 106–128. Read *May* 11, 1857.]

[Abstract. *Proc. Camb. Phil. Soc.* Vol. I, pp. 181–2.]

In a paper "On the Numerical Calculation of a class of Definite Integrals and Infinite Series" printed in the ninth volume of the *Cambridge Philosophical Transactions*, the author succeeded in putting the integral

$$\int_0^\infty \cos\frac{\pi}{2}(w^3 - mw)\,dw$$

under a form which admits of extremely easy numerical calculation when m is large, whether positive or negative. The integral is obtained in the first instance under the form of circular functions for m positive, or an exponential for m negative, multiplied by series according to descending powers of m. These series, which are at first convergent, though ultimately divergent, have arbitrary constants as coefficients, the determination of which is all that remains to complete the process. From the nature of the series, which are applicable only when m is large, or when it is an imaginary quantity with a large modulus, the passage from a large positive to a large negative value of m cannot be made through zero, but only by making m imaginary and altering its amplitude by π. The author succeeded in determining directly the arbitrary constants for m positive, but not for m negative. It was found that if, in the analytical expression applicable in the case of m positive, $-m$ were written for m, the result would become correct on throwing away the part involving an exponential with a positive index. There was nothing however to show *à priori* that this process was legitimate, nor, if it were, at what value of the amplitude of m a change in the analytical expression ought to be made, although the occurrence of radicals in the descending and ultimately divergent series, which did not occur in ascending convergent series by which the function might always be expressed, showed that some change analogous to the change of sign of a radical ought to be made in passing through some values of the amplitude of the variable m. The method which the author applied to this function is of very general application, but is subject throughout to the same difficulty.

In the present paper the author has resumed the subject, and has pointed out the character by which the liability to discontinuity in the arbitrary

satisfy, are expressed in such a manner as to involve only as many unknown constants as correspond to the degree of the equation.

Divergent series are usually divided into two classes, according as the terms are regularly positive, or alternately positive and negative. But according to the view here taken, series of the former kind appear as singularities of the general case of divergent series proceeding according to powers of an imaginary variable, as indeterminate forms in passing through which a discontinuity of analytical expression takes place, analogous to a change of sign of a radical.

In a paper "On the Numerical Calculation of a class of Definite Integrals and Infinite Series," printed in the ninth volume of the *Transactions* of this Society, I succeeded in developing the integral $\int_0^\infty \cos\frac{\pi}{2}(w^3 - mw)\,dw$ in a form which admits of extremely easy numerical calculation when m is large, whether positive or negative, or even moderately large. The method there followed is of very general application to a class of functions which frequently occur in physical problems. Some other examples of its use are given in the same paper; and I was enabled by the application of it to solve the problem of the motion of the fluid surrounding a pendulum of the form of a long cylinder, when the internal friction of the fluid is taken into account*.

These functions admit of expansion, according to ascending powers of the variables, in series which are always convergent, and which may be regarded as defining the functions for all values of the variable real or imaginary, though the actual numerical calculation would involve a labour increasing indefinitely with the magnitude of the variable. They satisfy certain linear differential equations, which indeed frequently are what present themselves in the first instance, the series, multiplied by arbitrary constants, being merely their integrals. In my former paper, to which the present may be regarded as a supplement, I have employed these equations to obtain integrals in the form of descending series multiplied by exponentials. These integrals, when once the arbitrary constants are determined, are exceedingly convenient for numerical calculation when the variable is large, notwithstanding

The determination of the arbitrary constants may be effected in two ways, numerically or analytically. In the former, it will be sufficient to calculate the function for one or more values of the variable from the ascending and descending series separately, and equate the results. This method has the advantage of being generally applicable, but is wholly devoid of elegance. It is better, when possible, to determine analytically the relations between the arbitrary constants in the ascending and descending series. In the examples to which I have applied the method, with one exception, this was effected, so far as was necessary for the physical problem, by means of a definite integral, which either was what presented itself in the first instance, or was employed as one form of the integral of the differential equation, and in either case formed a link of connexion between the ascending and the descending series. The exception occurs in the case of Mr Airy's integral for m negative. I succeeded in determining the arbitrary constants in the divergent series for m positive; but though I was able to obtain the correct result for m negative, I had to profess myself (p. 177, [343]) unable to give a satisfactory demonstration of it.

But though the arbitrary constants which occur as coefficients of the divergent series may be completely determined for real values of the variable, or even for imaginary values with their amplitudes lying between restricted limits, something yet remains to be done in order to render the expression by means of divergent series analytically perfect. I have already remarked in the former paper (p. 176, [342]) that inasmuch as the descending series contain radicals which do not appear in the ascending series, we may see, *a priori*, that the arbitrary constants must be discontinuous. But it is not enough to know that they must be discontinuous; we must also know where the discontinuity takes place, and to what the constants change. Then, and not till then, will the expressions by descending series be complete, inasmuch as we shall be able to use them for all values of the amplitude of the variable.

I have lately resumed this subject, and I have now succeeded

becomes thereby *to a certain extent* illusory; and thus it is that analysis gets over the apparent paradox of furnishing a discontinuous expression for a continuous function. It will be found that the expressions by divergent series will thus acquire all the requisite generality, and that though applied without any restriction as to the amplitude of the variable they will contain only as many unknown constants as correspond to the degree of the differential equation. The determination, among other things, of the constants in the development of Mr Airy's integral will thus be rendered complete *.

---

1. Before proceeding to more difficult examples, it will be well to consider a comparatively simple function, which has been already much discussed. As my object in treating this function is to facilitate the comprehension of methods applicable to functions of much greater complexity, I shall not take the shortest course, but that which seems best adapted to serve as an introduction to what is to follow.

Consider the integral

$$u = 2\int_0^\infty e^{-x^2} \sin 2ax\,dx = \frac{2a}{1} - \frac{(2a)^3}{2\,.\,3} + \frac{(2a)^5}{3\,.\,4\,.\,5} \;\ldots\ldots(1).$$

[* The considerations developed in this memoir form the complement of Section III of the memoir "On the Critical Values of the Sums of Periodic Series" (*Camb. Phil. Trans.* 1847; *ante* Vol. I, p. 279), in which the cause of discontinuity in the values of series and integrals was traced to infinitely slow convergence at the place of the sudden change. Here the converse phenomenon of discontinuous changes in the constants multiplying the semi-convergent series, which together represent a continuous function, is traced to the same kind of origin, in the domain however of the complex variable,—namely (p. 103) the multiplier of a series can change discontinuously only in crossing places at which one of the other series involved in the expression of the function loses its semi-convergence.

The formulae for the Bessel functions, employed here (pp. 100—103) as an illustration, were again obtained eleven years later by H. Hankel, in his memoir on the cylinder functions, *Math. Annalen*, Vol. I, Dec. 1868, p. 498, with the same explanation of the discontinuities of their constants. At that time the method of complex contour integration, developed by Riemann in his fundamental memoir

The integral and the series are both convergent for all values of a, and either of them completely defines u for all values real or imaginary of a. We easily find from either the integral or the series

$$\frac{du}{da} + 2au = 2 \quad \text{.....(2).}$$

This equation gives, if we observe that $u = 0$ when $a = 0$,

$$u = 2e^{-a^2}\int_0^a e^{a^2}da = 2e^{-a^2}\left\{a + \frac{a^3}{1.3} + \frac{a^5}{1.2.5} + \frac{a^7}{1.2.3.7} + \dots\right\}\dots(3).$$

This integral or series like the former gives a determinate and unique value to u for any assigned value of a real or imaginary. Both series, however, though ultimately convergent, begin by diverging rapidly when the modulus of a is large. For the sake of brevity I shall hereafter speak of an imaginary quantity simply as large or small when it is meant that its modulus is large or small.

2. In order to obtain u in a form convenient for calculation when a is large, let us seek to express u by means of a descending series. We see from (2) that when the real part of a^2 is positive, the most important terms of the equation are $2au$ and 2, and the leading term of the development is a^{-1}. Assuming a series with arbitrary indices and coefficients, and determining them so as to satisfy the equation, we readily find

$$u = \frac{1}{a} + \frac{1}{2a^3} + \frac{1.3}{2^2a^5} + \dots$$

This series can be only a particular integral of (2), since it wants an arbitrary constant. To complete the integral we must add the complete integral of

$$\frac{du}{da} + 2au = 0,$$

whence we get for the complete integral of (2)

$$u = Ce^{-a^2} + \frac{1}{a} + \frac{1}{2a^3} + \frac{1.3}{2^2a^5} + \frac{1.3.5}{2^3a^7} + \dots \text{.....(4).}$$

This expression might have been got at once from (3) by integration by parts. It remains to determine the arbitrary

$C = 0$, in which case the even function disappears. But since, as we shall presently see, the value of C is not zero, it must change sign with a. Let

$$a = \rho\,(\cos\theta + \sqrt{-1}\sin\theta).$$

Since in the application of the series (4) it is supposed that ρ is large, we must suppose a to change sign by a variation of θ, which must be increased or diminished (suppose increased) by π. Hence, if we knew what C was for a range π of θ, suppose from $\theta = \alpha$ to $\theta = \alpha + \pi$, we should know at once what it was from $\theta = \alpha + \pi$ to $\theta = \alpha + 2\pi$, which would be sufficient for our purpose, since we may always suppose the amplitude of a included in the range α to $\alpha + 2\pi$, by adding, if need be, a positive or negative multiple of 2π, which as appears from (1) or (3) makes no difference in the value of u.

4. When ρ is large the series (4) is at first rapidly convergent, but be ρ ever so great it ends by diverging with increasing rapidity. Nevertheless it may be employed in calculation provided we do not push the series too far, but stop before the terms get large again. To show in a general way the legitimacy of this, we may observe that if we stop with the term

$$\frac{1\,.\,3\,.\,5\ldots(2i-1)}{2^i a^{2i+1}},$$

the value of u so obtained will satisfy exactly, not (2), but the differential equation

$$\frac{du}{da} + 2au = 2 - \frac{1\,.\,3\ldots(2i+1)}{2^i a^{2i+2}} \quad\ldots\ldots\ldots\ldots\ldots(5).$$

Let u_0 be the true value of u for a large value of a_0 of a, and suppose that we pass from a_0 to another large value of a keeping the modulus of a large all the while. Since u ought to satisfy (2), we ought to have

$$u = u_0 + 2e^{-a^2}\int_{a_0}^{a} e^{a}\,da,$$

whereas since our approximate expression for u actually satisfies (5) we actually have, putting A_i for the last term,

be very small. Hence in general A_i may be neglected in comparison with (2), and we may use the expression (4), though we stop after $i+1$ terms of the series, as a near approximation to u.

5. But to this there is an important restriction, to understand which more readily it will be convenient to suppose the integration from a_0 to a performed, first by putting

$$da = (\cos\theta + \sqrt{-1}\sin\theta)\,d\rho,$$

and integrating from ρ_0 to ρ, θ remaining equal to θ_0, and then

$$da = \rho\,(-\sin\theta + \sqrt{-1}\cos\theta)\,d\theta,$$

and integrating from θ_0 to θ, ρ remaining unchanged. This is allowable, since u is a finite, continuous, and determinate function of a, and therefore the mode in which ρ and θ vary when a passes from its initial value a_0 to its final value a is a matter of indifference. The modulus of e^{a^2} will depend on the real part $\rho^2\cos 2\theta$ of the index. Now should $\cos 2\theta$ become a maximum within the limits of integration, we can no longer neglect A_i in the integration. For however great may be the value previously assigned to i, the quantity $\rho^{-2i-1}e^{\rho^2\cos 2\theta}$ will become, for values of θ comprised within the limits of integration, infinitely great, when ρ is infinitely increased, compared with the value of $e^{\rho^2\cos 2\theta}$ at either limit. And though the modulus of the quantity $2e^{a^2}$ under the integral sign will become far greater still, inasmuch as it does not contain the factor ρ^{-2i-1}, yet as the mutual destruction of positive and negative parts may take place quite differently in the two integrals $\int 2e^{a^2}da$ and $\int A_i e^{a^2}da$, we can conclude nothing as to their relative importance.

6. Now $\cos 2\theta$ will continually increase or decrease from one limit to the other, or else will become a maximum, according as the two limits θ_0 and θ lie in the same interval 0 to π or π to 2π, or else lie one in one of the two intervals and the other in the other. Hence we may employ the expression (4), with an invariable value of C yet to be determined, so long as $0 < \theta < \pi$, and we may employ the expression obtained by writing C' for C so long as $\pi < \theta < 2\pi$, but we must not pass from one interval to the

expression (4) generally applicable, it will be sufficient to change the sign of the constant whenever θ passes through zero or a multiple of π.

7. We may arrive at the same conclusion in another way, which will be of more general or at least easier application, as not involving the integration of the differential equation.

The modulus of the general term (Art. 4) of the series (4), expressed by means of the function Γ, is

$$\frac{\Gamma(i+\frac{1}{2})}{\Gamma(\frac{1}{2})\,\rho^{2i+1}}.$$

Suppose i very large. Employing the formula

$$\Gamma(x+1)=\sqrt{2\pi x}\left(\frac{x}{e}\right)^x, \text{ nearly, when } x \text{ is large,}$$

observing that $\Gamma(\frac{1}{2})=\pi^{\frac{1}{2}}$, and calling the modulus μ_i, we find

$$\mu_i = 2^{\frac{1}{2}}(i-\tfrac{1}{2})^i\, e^{-i+\frac{1}{2}}\rho^{-2i-1},$$

which, since $(i+c)^i = i^i e^c$, nearly, becomes

$$\mu_i = 2^{\frac{1}{2}} i^i e^{-i} \rho^{2i-1} \quad \text{.....(7).}$$

We easily get, either from this expression or from the general term,

$$\frac{\mu_{i+1}}{\mu_i} = \frac{i}{\rho^2}, \text{ nearly.} \quad \text{.....(8).}$$

Hence when ρ is large the ratio of consecutive moduli becomes very nearly equal to unity for a great number of terms together, about where the modulus is a minimum. To find approximately the minimum modulus μ, we must put $i=\rho^2$ in (7), which gives

$$\mu = 2^{\frac{1}{2}}\rho^{-1}e^{-\rho^2} \quad \text{.....(9).}$$

If we knew precisely at what term it would be best to stop, the expression for μ would be a measure of the uncertainty to which we were liable in using the series (4) directly, that is, without any transformation. For although it is clear that we must stop *somewhere about* the term with a minimum modulus, in order that the differential equation (5) which our function really satisfies

n terms, the sum of the moduli of these n nearly equal term will be

$$2^{\frac{1}{2}} n\rho^{-1}e^{-\rho^2},$$

nearly. It seems as if n must increase with i, but not so fast as If we suppose that it is of the form $ki^{\frac{1}{2}}$ or $k\rho$, the sum of th n terms will be a quantity of the order $e^{-\rho^2}$. But even if increased as any power p of i, however great, still the sum c the n terms would be a quantity of the order $\rho^{2p-1}e^{-\rho^2}$, which whe ρ was infinitely increased would become infinitely small in com parison with the modulus $e^{-\rho^2\cos 2\theta}$ of the term multiplied by in (4), provided θ had any given value differing from zero or multiple of π. Hence if θ have any value lying between α an $\pi-\alpha$, or else between $\pi+\alpha$ and $2\pi-\alpha$, where α is a smal positive quantity which in the end may be made as small as w please, the quantity C in (4) cannot pass from one of its values t another without rendering the function u discontinuous, which i is not. But when $\theta=0$ or $=\pi$, the term $Ce^{-\alpha^2}$ becomes merged i the vagueness with which, in this case, the divergent series define the function. Hence we arrive in a way quite different from tha of Art. 5 at the conclusions enunciated in Art. 6.

8. Nor is this all. When the terms of a regular series ar alternately positive and negative, the series may be converted b the formulæ of finite differences into others which converge rapidly In the present case the terms are not simply positive and negativ alternately, except when θ is an odd multiple of $\frac{1}{2}\pi$, but the sam methods will apply with the proper modification. Suppose tha we sum the series (4) directly as far as terms of the order $i-$ inclusive. Omitting the common factor $e^{-(2i+1)\theta\sqrt{-1}}$, which may b restored in the end, we have for the rest of the series

$$\mu_i + e^{-2\theta\sqrt{-1}}\mu_{i+1} + e^{-4\theta\sqrt{-1}}\mu_{i+2} + \dots$$

If we denote by D or $1+\Delta$ the operation of passing from μ_i t μ_{i+1}, and separate symbols of operation, this becomes

$$(1 + e^{-2\theta\sqrt{-1}}D + e^{-4\theta\sqrt{-1}}D^2 + \dots)\,\mu_i,$$

or $\{1-(1+\Delta)e^{-2\theta\sqrt{-1}}\}^{-1}\dots$

or, putting q for $(2 \sin \theta)^{-1}$, to

$$qe^{\left(\theta - \frac{\pi}{2}\right)\sqrt{-1}} \mu_i + q^2 e^{-\pi\sqrt{-1}} \Delta\mu_i + q^3 e^{-\left(\frac{3\pi}{2} + \theta\right)\sqrt{-1}} \Delta^2\mu_i + \dots$$

Now if ρ be very large, and μ_i belong to the part of the series where the moduli of consecutive terms are nearly equal, the successive differences $\Delta\mu_i, \Delta^2\mu_i, \dots$ will decrease with great rapidity. Hence if θ have any given value different from zero or a multiple of π, by taking ρ sufficiently great, we may transform the series about where it ceases to converge into one which is at first rapidly convergent, and thus a quantity which may be taken as a measure of the remaining uncertainty will become incomparably smaller even than μ, much more, incomparably smaller than the modulus of e^{-a^2}. But if $\theta = 0$ or $= \pi$, the above transformation fails, since q becomes infinite. In this case if we want to calculate u closer than to admit of the uncertainty to which we are liable, knowing only that we must stop *somewhere about* the place where the series begins to diverge after having been convergent, we must have recourse to the ascending series (1) or (3), or to some perfectly distinct method. The usual method by which Σu_x is made to depend on $\int u_x dx$ would evidently fail, in consequence of the divergence of the integral.

9. In applying *practically* the transformation of the last article to the summation of the series (4), it would not usually, when ρ was very large, be necessary to go as far as the part of the series where the moduli of consecutive terms are nearly equal. It would be sufficient to deduct l, $2l \dots$ from the logarithms of μ_{i+1}, $\mu_{i+2} \dots$, where l is nearly equal to the mean increment of the logarithms at that part of the series, to associate the factor f whose logarithm is l with the symbol D, and take the differences of the numbers

$$\mu_i, \ f^{-1}\mu_{i+1}, \ f^{-2}\mu_{i+2}, \text{ etc.}$$

However, my object leads me to consider, not the actual summation of the series, but the theoretical possibility of summation, and consequent interpretation of the equation (4).

10. The mode of discontinuity of the constant C having been

whence, putting $a=\infty$, we have $C=\sqrt{-1}\pi^{\frac{1}{2}}$. Hence we get for the general expression for C in (4),

$$\left.\begin{array}{l} C=\sqrt{-1}\pi^{\frac{1}{2}}, \quad \text{when } 0<\theta<\pi, \\ C=-\sqrt{-1}\pi^{\frac{1}{2}} \text{ when } \pi<\theta<2\pi; \end{array}\right\} \ldots\ldots\ldots\ldots(10),$$

and therefore from (3) and (4)

$$2e^{a^2}\int_0^a e^{a^2}da = \pm\sqrt{-1}\pi^{\frac{1}{2}}e^{a^2}+\frac{1}{a}+\frac{1}{2a^3}+\frac{1.3}{2^2a^5}+\frac{1.3.5}{2^3a^7}+\ldots\ldots(11),$$

the sign being + or − according as θ, the amplitude of a, is comprised within the limits 0 and π, or π and 2π.

Writing $a\sqrt{-1}$ for a in (11), which comes to altering the origin of θ by $\frac{1}{2}\pi$, we find

$$2e^{a^2}\int_0^a e^{-a^2}da = \pm\pi^{\frac{1}{2}}e^{a^2}-\frac{1}{a}+\frac{1}{2a^3}-\frac{1.3}{2^2a^5}+\frac{1.3.5}{2^3a^7}-\ldots\ldots(12),$$

the sign being + or − according as the amplitude of a lies within the limits $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$, or $\frac{1}{2}\pi$ and $\frac{3}{2}\pi$. It is worthy of remark that in this expression the transcendental quantity $\pi^{\frac{1}{2}}$ appears as a true radical, admitting of the double sign.

Two cases of the integral $\int_0^a e^{a^2}da$ occur in actual investigations, namely when $\theta=\frac{1}{2}\pi$, when the integral leads to $\int_0^t e^{t^2}dt$ which occurs in the theory of probabilities, and when $\theta=\frac{1}{4}\pi$, when it leads to Fresnel's integrals $\int_0^s \cos(\frac{1}{2}\pi s^2)\,ds$ and $\int_0^s \sin(\frac{1}{2}\pi s^2)\,ds$. In the latter case the expression (11) is equivalent to the development of these integrals which has been given by M. Cauchy.

11. If in equation (11) we put $a=\rho(\cos\theta\pm\sqrt{-1}\sin\theta)$, where θ is a small positive quantity, and after equating the rea

The expression which appears on the second side of this equation may be regarded as a *singular value* of the sum of the series

$$\frac{1}{a}+\frac{1}{2a^3}+\frac{1.3}{2^2a^5}+\frac{1.3.5}{2^3a^7}+\ldots\ldots\ldots\ldots(14),$$

a series which when θ vanishes *takes the form* of the first member of the equation. The equivalent of the series for general values of the variable is given, not by (13), but by (11). It may be remarked that the singular value is the mean of the general values for two infinitely small values of θ, one positive and the other negative.

These results, to which we are led by analysis, may be compared with the known theory of periodic series. If $f(x)$ be a finite function of x, the value of which changes abruptly from a to b as x increases through the value c, a quantity lying between 0 and π, and $f(x)$ be expanded between the limits 0 and π in a series of sines of multiples of x, and if $\phi(n, x)$ be the sum of n terms of the series, the value of $\phi(n, x)$ for an infinitely large value of n and a value of x infinitely near to c is indeterminate, like that of the fraction

$$\frac{(x+y)^2+x-y}{(x-y)^2+x+y},$$

which takes the form $\frac{0}{0}$ when x and y vanish, but of which the limiting value is wholly indeterminate if x and y are independent. We may enquire, if we please, what is the limit of the fraction when x first vanishes and then y, or the limit when y first vanishes and then x, for each of these has a perfectly clear and determinate signification. In the former case we have, calling the fraction $\psi(x, y)$,

$$\lim_{y=0}\lim_{x=0}\psi(x, y)=\lim_{y=0}\frac{y^2-y}{y^2+y}=-1;$$

the latter

$$\lim_{x=0}\lim_{y=0}\psi(x, y)=\lim_{x=0}\frac{x^2+x}{x^2+x}=1.$$

So in the case of the periodic series if we denote by ξ a small positive quantity

but we know that

$$\lim._{n=\infty} \lim._{\xi=0} \phi(n, c \pm \xi) = \lim._{n=\infty} \phi(n, c) = \tfrac{1}{2}(a+b).$$

Similarly in the case of the series (14) if we denote its sum by $\chi(a) = \varpi(\rho, \theta)$, and use the term *limit* in an extended sense, so as to understand by $\lim._{\rho=\infty} F(\rho)$ a function of ρ to which $F(\rho)$ may be regarded as equal when ρ is large enough, and if we suppose θ to be a small positive quantity, we have from (11)

$$\lim._{\theta=0} \lim._{\rho=\infty} \varpi(\rho, \theta) = \lim._{\theta=0} \{2e^{-a^2} \int_0^a e^{a^2} da - \sqrt{-1}\,\pi^{\frac{1}{2}} e^{-a^2}\}$$

$$= 2e^{-\rho^2} \int_0^\rho e^{\rho^2} d\rho - \sqrt{-1}\,\pi^{\frac{1}{2}} e^{-\rho^2};$$

$$\lim._{\theta=0} \lim._{\rho=\infty} \varpi(\rho, -\theta) = \lim._{\theta=0} \{2e^{-a^2} \int_0^a e^{a^2} da + \sqrt{-1}\,\pi^{\frac{1}{2}} e^{a^2}\}$$

$$= 2e^{-\rho^2} \int_0^\rho e^{\rho^2} d\rho + \sqrt{-1}\,\pi^{\frac{1}{2}} e^{-\rho^2},$$

whereas equation (13) may be expressed by

$$\lim._{\rho=\infty} \lim._{\theta=0} \varpi(\rho, \pm\theta) = \lim._{\rho=\infty} \chi(\rho) = 2e^{-\rho^2} \int_0^\rho e^{\rho^2} d\rho.$$

There is however this difference between the two cases, that in the case of the periodic series the series whose general term is $\Delta\phi(n, c)$ is convergent, and may be actually summed to any assigned degree of accuracy, whereas the series (13), though at first convergent, is ultimately divergent; and though we know that we must stop somewhere about the least term, that alone does not enable us to find the sum, except subject to an uncertainty comparable with $e^{-\rho^2}$. Unless therefore it be possible to apply to the series (13) some transformation rendering it capable of summation to a degree of accuracy incomparably superior to this, the equation (13) must be regarded as a mere symbolical result. We might indeed *define* the sum of the ultimately divergent series (13) to mean the sum taken to as many terms as should *make* the equation (13) true, and express that condition in a manner which would not require the quantity taken to denote the number of

12. In order still further to illustrate the subject, before going on to the actual application of the principles here established, let us consider the function defined by the equation

$$u = 1 + \frac{1}{2}x - \frac{1.1}{2.4}x^2 + \frac{1.1.3}{2.4.6}x^3 - \ldots\ldots\ldots\ldots(15).$$

Suppose that we have to deal with such values only of the imaginary variable x as have their moduli less than unity. For such values the series (15) is convergent, and the equation (15) assigns a determinate and unique value to u. Now we happen to know that the series is the development of $(1+x)^{\frac{1}{2}}$. But this function admits of one or other of the following developments according to descending powers of x:—

$$u = x^{\frac{1}{2}} + \frac{1}{2}x^{-\frac{1}{2}} - \frac{1.1}{2.4}x^{-\frac{3}{2}} + \frac{1.1.3}{2.4.6}x^{-\frac{5}{2}} - \ldots\ldots\ldots(16),$$

$$u = -x^{\frac{1}{2}} - \frac{1}{2}x^{-\frac{1}{2}} + \frac{1.1}{2.4}x^{-\frac{3}{2}} - \frac{1.1.3}{2.4.6}x^{-\frac{5}{2}} + \ldots\ldots\ldots(17).$$

Let $x = \rho(\cos\theta + \sqrt{-1}\sin\theta)$, and let $x^{\frac{1}{2}}$ denote that square root of x which has $\frac{1}{2}\theta$ for its amplitude. Although the series (16), (17) are divergent when $\rho < 1$, they may in general, for a given value of θ, be employed in actual numerical calculation, by subjecting them to the transformation of Art. 8, provided ρ do not differ too much from 1. The greater be the accuracy required, θ being given, the less must ρ differ from 1 if we would employ the series (16) or (17) in place of (15). It remains to be found which of these series must be taken.

If θ lie between $(2i-1)\pi + \alpha$ and $(2i+1)\pi - \alpha$, where i is any positive or negative integer or zero, and α a small positive quantity which in the end may be made as small as we please, either series (16) or (17) may by the method of Art. 8 be converted into another, which is at first sufficiently convergent to give u with a sufficient degree of accuracy by employing a finite number only of terms. If m terms be summed directly, and in the formula of Art. 8 the nth difference be the last which yields significant

an odd multiple of π we may have to pass from one of the two series to the other. Now when θ is increased by 2π the series (16) or (17) changes sign, whereas (15) remains unchanged. Therefore in calculating u for two values of θ differing by 2π we must employ the two series (16) and (17), one in each case. Hence we must employ one of the series from $\theta = -\pi$ to $\theta = \pi$, the other from $\theta = \pi$ to $\theta = 3\pi$, and so on; and therefore if we knew which series to take for some one value of x everything would be determined.

Now when $\rho = 1$ the series (15) becomes identical with (16) when θ has the particular value 0. Hence (16) and not (17) gives the true value of u when $-\pi < \theta < \pi$.

13. Let ρ, θ be the polar co-ordinates of a point in a plane, O the origin, C a circle described round O with radius unity, S the point determined by $x = -1$, that is, by $\rho = 1$, $\theta = \pi$. To each value of x corresponds a point in the plane; and the restriction laid down as to the moduli of x confines our attention to points within the circle, to each of which corresponds a determinate value of u. If P_0 be any point in the plane, either within the circle or not, and a moveable point P start from P_0, and after making any circuit, without passing through S, return to P_0 again, the function $(1+x)^{\frac{1}{2}}$ will regain its primitive value u_0, or else become equal to $-u_0$, according as the circuit excludes or includes the point S, which for the present purpose may be called a *singular point*. Suppose that we wished to tabulate u, using when possible the divergent series (16) in place of the convergent series (15). For a given value of θ, in commencing with small values of ρ we should have to begin with the series (15), and when ρ became large enough we might have recourse to (16). Let OP be the smallest value of ρ for which the series (16) may be employed; for which, suppose, it will give u correctly to a certain number of decimal places. The length OP will depend upon θ, and the locus of P will be some curve, symmetrical with respect to the diameter through S. As θ increases the curve will gradually approach the

12. In order still further to illustrate the subject, before going on to the actual application of the principles here established, let us consider the function defined by the equation

$$u = 1 + \frac{1}{2}x - \frac{1.1}{2.4}x^2 + \frac{1.1.3}{2.4.6}x^3 - \ldots\ldots\ldots\ldots(15).$$

Suppose that we have to deal with such values only of the imaginary variable x as have their moduli less than unity. For such values the series (15) is convergent, and the equation (15) assigns a determinate and unique value to u. Now we happen to know that the series is the development of $(1+x)^{\frac{1}{2}}$. But this function admits of one or other of the following developments according to descending powers of x:—

$$u = x^{\frac{1}{2}} + \frac{1}{2}x^{-\frac{1}{2}} - \frac{1.1}{2.4}x^{-\frac{3}{2}} + \frac{1.1.3}{2.4.6}x^{-\frac{5}{2}} - \ldots\ldots\ldots(16),$$

$$u = -x^{\frac{1}{2}} - \frac{1}{2}x^{-\frac{1}{2}} + \frac{1.1}{2.4}x^{-\frac{3}{2}} - \frac{1.1.3}{2.4.6}x^{-\frac{5}{2}} + \ldots\ldots\ldots(17).$$

Let $x = \rho(\cos\theta + \sqrt{-1}\sin\theta)$, and let $x^{\frac{1}{2}}$ denote that square root of x which has $\frac{1}{2}\theta$ for its amplitude. Although the series (16), (17) are divergent when $\rho < 1$, they may in general, for a given value of θ, be employed in actual numerical calculation, by subjecting them to the transformation of Art. 8, provided ρ do not differ too much from 1. The greater be the accuracy required, θ being given, the less must ρ differ from 1 if we would employ the series (16) or (17) in place of (15). It remains to be found which of these series must be taken.

If θ lie between $(2i-1)\pi + \alpha$ and $(2i+1)\pi - \alpha$, where i is any positive or negative integer or zero, and α a small positive quantity which in the end may be made as small as we please, either series (16) or (17) may by the method of Art. 8 be converted into another, which is at first sufficiently convergent to give u with a sufficient degree of accuracy by employing a finite number only of terms. If m terms be summed directly, and in the formula of Art. 8 the nth difference be the last which yields significant figures, the number of terms actually employed in some way or

The integral in a form adapted for calculation when x is large, obtained by the method of my former paper, is

$$\left.\begin{aligned} u = Cx^{-\frac{1}{4}}e^{-2x^{\frac{3}{2}}}&\left\{1 - \frac{1.5}{1.144x^{\frac{3}{2}}} + \frac{1.5.7.11}{1.2.144^2x^3} - \frac{1.5.7.11.13.17}{1.2.3.144^3x^{\frac{9}{2}}} + \ldots\right\} \\ + Dx^{-\frac{1}{4}}e^{2x^{\frac{3}{2}}}&\left\{1 + \frac{1.5}{1.144x^{\frac{3}{2}}} + \frac{1.5.7.11}{1.2.144^2x^3} + \frac{1.5.7.11.13.17}{1.2.3.144^3x^{\frac{9}{2}}} + \ldots\right\} \end{aligned}\right\} \quad \ldots\ldots\ldots(20).$$

The constants C, D must however be discontinuous, since otherwise the value of u determined by this equation would not recur, as it ought, when the amplitude of x is increased by 2π. We have now first to ascertain the mode of discontinuity of these constants, secondly, to find the two linear relations which connect A, B with C, D.

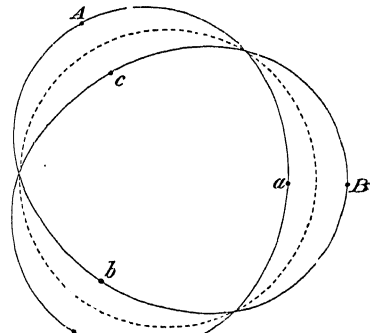
Let the equation (20) be denoted for shortness by

$$u = Cx^{-\frac{1}{4}}f_1(x) + Dx^{-\frac{1}{4}}f_2(x) \quad \ldots\ldots\ldots\ldots\ldots(21);$$

and let $f(x)$, when we care only to express its dependence on the amplitude of x, be denoted by $F(\theta)$. We may notice that

$$F_1(\theta + \tfrac{2}{3}\pi) = F_2(\theta); \qquad F_2(\theta + \tfrac{2}{3}\pi) = F_1(\theta) \ldots\ldots(22).$$

15. In equation (21), let that term in which the real part of the index of the exponential is positive be called the *superior*, and the other the *inferior* term. In order to represent to the eye the existence and progress of the functions $f_1(x)$, $f_2(x)$ for different values of θ, draw a circle with any radius, and along a radius vector inclined to the prime radius at the variable angle θ take two



same series (16) for the calculation of u for values of x having amplitudes $\pi - \beta$, $\pi + \beta$, corresponding to points P, P', we should get for the value of u at P' that into which the value of u at P passes continuously when we travel from P to P' *outside* the point S, which as we have seen is *minus* the true value, the latter being defined to be that into which the value of u at P passes continuously when we travel from P to P' *inside* the point S.

In the case of the simple function at present under consideration, it would be an arbitrary restriction to confine our attention to values of x having moduli less than unity, nor would there be any advantage in using the divergent series (16) rather than the convergent series (15). But in the example first considered we have to deal with a function which has a perfectly determinate and unique value for all values of the variable a, and there is the greatest possible advantage in employing the descending series for large values of ρ, though it is ultimately divergent. In the case of this function there are (to use the same geometrical illustration as before) as it were two singular points at infinity, corresponding respectively to $\theta = 0$ and $\theta = \pi$.

14. The principles which are to guide us having been now laid down, there will be no difficulty in applying them to other cases, in which their real utility will be perceived. I will now take Mr Airy's integral, or rather the differential equation to which it leads, the treatment of which will exemplify the subject still better. This equation, which is No. 11 of my paper "On the Numerical Calculation, etc.," becomes on writing u for U, $-3x$ for n

$$\frac{d^2u}{dx^2} - 9xu = 0 \quad \text{.....(18).}$$

The complete integral of this equation in ascending series, obtained in the usual way, is

$$\left.\begin{aligned} u = A &\left\{1 + \frac{9x^3}{2.3} + \frac{9^2x^6}{2.3.5.6} + \frac{9^3x^9}{2.3.5.6.8.9} + \dots\right\} \\ + B &\left\{x + \frac{9x^4}{\ } + \frac{9^2x^7}{\ } + \frac{9^3x^{10}}{\ } \dots\right\} \end{aligned}\right\} \text{....(19).}$$

The integral in a form adapted for calculation when x is large, obtained by the method of my former paper, is

$$\left.\begin{aligned} u = Cx^{-\frac{1}{4}}e^{-2x^{\frac{3}{2}}}&\left\{1 - \frac{1.5}{1.144x^{\frac{3}{2}}} + \frac{1.5.7.11}{1.2.144^2x^3} - \frac{1.5.7.11.13.17}{1.2.3.144^3x^{\frac{9}{2}}} + \ldots\right\} \\ + Dx^{-\frac{1}{4}}e^{2x^{\frac{3}{2}}}&\left\{1 + \frac{1.5}{1.144x^{\frac{3}{2}}} + \frac{1.5.7.11}{1.2.144^2x^3} + \frac{1.5.7.11.13.17}{1.2.3.144^3x^{\frac{9}{2}}} + \ldots\right\} \end{aligned}\right\} \quad \ldots\ldots\ldots(20).$$

The constants C, D must however be discontinuous, since otherwise the value of u determined by this equation would not recur, as it ought, when the amplitude of x is increased by 2π. We have now first to ascertain the mode of discontinuity of these constants, secondly, to find the two linear relations which connect A, B with C, D.

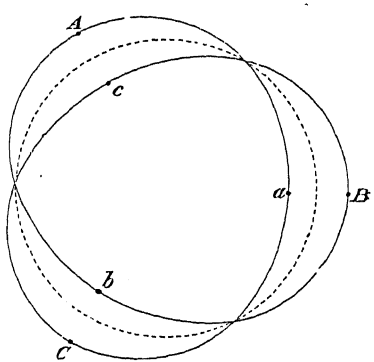
Let the equation (20) be denoted for shortness by

$$u = Cx^{-\frac{1}{4}}f_1(x) + Dx^{-\frac{1}{4}}f_2(x) \quad \ldots\ldots\ldots\ldots\ldots(21);$$

and let $f(x)$, when we care only to express its dependence on the amplitude of x, be denoted by $F(\theta)$. We may notice that

$$F_1(\theta + \tfrac{2}{3}\pi) = F_2(\theta); \qquad F_2(\theta + \tfrac{2}{3}\pi) = F_1(\theta) \ldots\ldots(22).$$

15. In equation (21), let that term in which the real part of the index of the exponential is positive be called the *superior*, and the other the *inferior* term. In order to represent to the eye the existence and progress of the functions $f_1(x)$, $f_2(x)$ for different values of θ, draw a circle with any radius, and along a radius vector inclined to the prime radius at the variable angle θ take two distances, measured respec-



or in other words proportional to $\cos \frac{3}{2}\theta$. For greater convenience suppose these distances moderately small compared with the radius. Consider first the function $F_1(\theta)$ alone. The curve will evidently have the form represented in the figure, cutting the circle at intervals of 120°, and running into itself after two complete revolutions. The equations (22) show that the curve corresponding to $F_2(\theta)$ is already traced, since $F_2(\theta) = F_1(\theta + 2\pi)$. If now we conceive the curve marked with the proper values of the constants C, D, it will serve to represent the complete integral of equation (18).

In marking the curve we may either assume the amplitude θ of x to lie in the interval 0 to 2π, and determine the values of C, D accordingly, or else we may retain the same value of C or D throughout as great a range as possible of the curve, and for that purpose permit θ to go beyond the above limits. The latter course will be found the more convenient.

16. We must now ascertain in what cases it is possible for the constant C or D to alter discontinuously as θ alters continuously. The tests already given will enable us to decide.

The general term of either series in (20), taken without regard to sign, is

$$\frac{1 \,.\, 5 \ldots (6i-5)(6i-1)}{1 \,.\, 2 \ldots i\,(144x^{\frac{3}{2}})^i};$$

and the modulus of this term, expressed by means of the function Γ, is

$$\frac{\Gamma(i+\frac{1}{6})\,\Gamma(i+\frac{5}{6})}{\Gamma(\frac{1}{6})\,\Gamma(\frac{5}{6})\,\Gamma(i+1)\,(4\rho^{\frac{3}{2}})^i},$$

which when i is very large becomes by the transformations employed in Art. 7, very nearly,

$$\sqrt{\frac{2\pi}{i}}\left(\frac{i}{e}\right)^i \div \Gamma(\tfrac{1}{6})\,\Gamma(\tfrac{5}{6})\,(4\rho^{\frac{3}{2}})^i.$$

Denoting this expression by u_i and putting for $\Gamma(\frac{1}{6})\,\Gamma(\frac{5}{6})$ its

whence for very large values of i

$$\frac{\mu_{i+1}}{\mu_i} = \frac{i}{4\rho^{\frac{3}{2}}} \quad \text{.....(24).}$$

For large values of ρ the moduli of several consecutive terms are nearly equal at the part of the series where the modulus is a minimum, and for the minimum modulus μ we have very nearly from (24), (23)

$$i = 4\rho^{\frac{3}{2}}, \qquad \mu = (2\pi i)^{-\frac{1}{2}} e^{-i} = (2\pi i)^{-\frac{1}{2}} e^{-4\rho^{\frac{3}{2}}}.$$

If the exponential in the expression for μ be multiplied by the modulus of the exponential in the superior term, the result will be

$$e^{-(4 \mp 2 \cos \frac{3}{2}\theta)\, \rho^{\frac{3}{2}}},$$

the sign $-$ or $+$ being taken according as $\cos \frac{3}{2}\theta$ is positive or negative. Hence even if the terms of the divergent series were all positive, the superior term would be defined by means of its series within a quantity incomparably smaller, when ρ is indefinitely increased, than the inferior term, except only when $\pm \cos \frac{3}{2}\theta = 1$, and in this case too and this alone are the terms of the divergent series in the superior term regularly positive. In no other case then can the coefficient of the inferior term alter discontinuously, and the coefficient of the other term cannot change so long as that term remains the superior term. Referring for convenience to the figure (Fig. 1), we see that it is only at the points a, b, c, at the middle of the portions of the curve which lie within the circle, that the coefficient belonging to the curve can change.

It might appear at first sight that we could have three distinct coefficients, corresponding respectively to the portions aAb, bBc, cCa of the curve, which would make three distinct constants occurring in the integral of a differential equation of the second order only. This however is not the case; and if we were to assign in the first instance three distinct constants to those three portions of the curve, they would be connected by an equation of condition.